Toric Varieties in a Nutshell

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Abstract

In algebraic geometry a variety is called toric if it has an embedded torus $(\mathbb{C}^*)^n$ whose Zariski closure is the variety itself. We focus on normal toric varieties since they can be studied from a combinatorial point of view. We introduce the basic theory and construct tools to compute explicit geometric information from combinatorics including smoothness, compactness, torus orbits, subvarieties and classes of divisors. We analyse the relation with GIT and resolution of singularities preserving the toric structure. We finish with an application of the theory to the calculation of partition functions or string theories over the singular varieties $\mathbf{V}(xy - z^{N_0}w^{N^1})$.

Contents

Introduction

In the last 20 years the field of toric geometry has experienced a huge development and it is not rare to find papers in algebraic geometry or mathematical physics where examples arise from toric varieties. The reason for this is that toric varieties can be approached in a combinatorial way, helping to develop a geometric intuition and making possible to compute examples of abstract concepts such as GIT quotients, classes of divisors or crepant resolutions. Many of these computations can be done algorithmically [BIP10a] and there is a software package available for Macaulay2, in particular to compute cohomology classes of toric bundles [BIP10b].

In mathematical physics toric varieties have become relevant in the theory of gauge linear sigma models (GLSM), as it is stated in [HKK$^+$03]:

> "In the absence of a superpotential, the set of supersymmetric ground states of the GLSM is a toric variety. Conversely, toric varieties can be described as the set of ground states of an appropriate gauge linear sigma model."

Moreover, many relevant examples of Mirror Symmetry and duality correspondence for Landau Ginzburg models [Cla08] are known for toric varieties. Furthermore, toric geometry is a powerful tool for the study of moduli spaces of representation of quivers [Cra08].

The main goal of this report is to present in a concise but complete way the basic constructions of toric varieties and establish procedures to obtain information about their geometric properties. There are many surveys on toric varieties but usually they just compute examples without actually explaining why those methods work. We try to justify all the steps, avoiding the most technical in the interest of brevity, but providing adequate references.

Briefly, a variety X is toric if it contains a torus $(\mathbb{C}^*)^n$ as a dense Zariski open subset such that the torus action on $(\mathbb{C}^*)^r$ extends to X. Toric varieties have an associated object called the fan which encodes all the one-parameter subgroups of the embedded torus. Fans are made of cones, one cone for each of the sets in a torus invariant open cover. However, fans can be defined independently of any toric variety and a normal toric variety can be recovered from its fan. When we restrict to normal toric varieties, this correspondence between fans and toric varieties is bijective up to isomorphism. This is proved in Chapter 2. Prior to this, in Chapter 1 we describe the basic definitions in detail and how to go from the fan to the variety and vice versa. Moreover, this chapter analyses morphisms between fans and equivariant morphisms between toric varieties and how they induce each other.

From Chapter 2 onwards we will restrict ourselves to normal toric varieties, once their correspondence with fans is proved. We also give an exhaustive criteria to determine when a variety is compact or smooth inspecting its fan and we establish a one-to-one correspondence between cones and torus-invariant orbits (the so called Orbit-Cone correspondence). The chapter finishes with a method to compute classes of divisors for normal toric varieties.

Chapter 3 introduces more advanced topics. After recalling briefly some Geometric Invariant Theory (GIT) we show how a normal toric variety is a GIT quotient of another toric variety (some Zariski open set of the affine space). This provides a version of the Orbit-Cone correspondence for subvarieties. The next chapter shows how any toric variety can be resolved algorithmically via toric resolutions in a finite number of steps, simplifying the more general result for varieties over algebraic closed fields of characteristic 0 in [Hir64] and [Hau03].

We finish with Chapter 5 where we show how to resolve the generalised conifold $\mathbf{V}(xy - z^{N_0}w^{N_1}) \subset \mathbb{C}^4$ via toric crepant resolutions[1]. We also give an algorithm to compute all the embedded bundles product of the resolution. This is needed in order to compute certain partition functions for the conifold, and it appears in [Nag10], [Sze08] and [GKM$^+$on].

Notation

All varieties, unless otherwise stated will be abstract algebraic varieties over $\mathbb{C}$, according to the following definition:

Definition 0.0.1. X is an **abstract variety** or just a **variety** if

$$X = \bigsqcup_{\alpha=1}^{k} V_\alpha \Big/ \sim,$$

where V_α are affine varieties and for all pairs α, β we have Zariski open sets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\beta\alpha} \colon V_{\beta\alpha} \cong V_{\alpha\beta}$ satisfying:

(i) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$,

(ii) $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\alpha}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$ for all α, β, γ,

where $a \sim b$ if and only if $a \in V_{\alpha\beta}, b \in V_{\beta\alpha}$ and $b = g_{\beta\alpha}(a)$, an equivalence relation. The **usual cover** will be $U_\alpha = \{[a] \in X \ : \ a \in V_\alpha\} \cong V_\alpha$.

We will assume without mention that all our varieties are **separated**, i.e. that the image of $X \hookrightarrow X \times X$ is a Zariski closed set. Standard facts from Algebra and Algebraic Geometry will be assumed, as presented in [AM69], [Har77] or the more accessible [Har95]. Standard facts from Geometric invariant theory will be assumed but the general theory will be recalled informally, [MFK94] is the standard reference.

In particular recall the following definitions:

Definition 0.0.2. A **lattice** $N \cong \mathbb{Z}^n$ is a free Abelian group of rank n. Its **dual lattice** is $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$.

Definition 0.0.3. A Weil divisor D is $\mathbb{Q}$-Cartier if some positive multiple of D is Cartier.

Definition 0.0.4. Given an irreducible variety X a **projective resolution of** X is a morphism $f \colon X' \to X$ such that:

[1]This variety often appears in the context of string theory and Gromov-Witten theory and it refers to the fact that it looks a like a 3-dimensional cone. The reason we use the term 'generalised' is to distinguish it from the particular case where $N_0 = N_1 = 1$ which is usually called just 'conifold'.

(i) X' is smooth and irreducible.

(ii) f is projective.

(i) f induces an isomorphism of varieties $f^{-1}(X \setminus X_{\text{sing}}) \cong X \setminus X_{\text{sing}}$.

All the constructions and theorems are for arbitrary lattices $N \cong \mathbb{Z}^n$, and some choice of pairing $\langle , \rangle$ with its dual lattice $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. However when it comes to examples, we will use $N = \mathbb{Z}^n$ and the usual pairing in $\mathbb{Z}^n$ to avoid confusion.

All the cones (Definition 1.1.3) will be **strongly convex rational polyhedral cones (scrapcs)**, unless otherwise stated and we will often refer to them simply as **cones**. This is not the case of dual cones, since duality does not preserve strong convexity (unless the dimension of the cones and the ambient lattice is the same).

For simplicity, the dimension of the fans (Definition 1.1.12) will be the same as the dimension of the lattice. This guarantees that the toric variety X_Σ of a fan Σ does not have torus factors (i.e. X_Σ is not the direct product of a toric variety and a torus). For fans with smaller dimension than the ambient lattice sometimes more work is needed, especially in Chapter 3.

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My main reference writing this paper has been [CLSar], although chapter 7 in [HKK$^+$03] and the expository article [Cox03] helped to get me started. The combinatorial results involving lattices and cones are taken from [Ful93] and they are stated without proof although precise references to the lemmas are provided. The most complicated theorems like the Orbit-Cone correspondence (Theorem 2.3.5), Cox's construction of toric varieties as GIT quotients (Theorem 3.2.6) and the torus resolution of singularities (theorem 4.0.12) are from [CLSar], although I have simplified some proofs or quoted technical parts which were lengthy. The physics background on the Gauge Linear Sigma Model can be found in [HKK$^+$03] or [CK99]. I make no claim of originality. The proofs of some statements (in particular theorems 2.4.8 and 3.3.1 and part of 2.1) are my own. The example with the resolution of $\mathbf{V}(xy - z^{N_0}w^{N_1})$, including the algorithm to count all the embedded curves is also my own work.

Chapter 1

Toric varieties and fans.

The purpose of this section is to establish the relation between toric varieties and fans. We start defining cones and fans as well as some of their properties. Then we show how we can construct a toric variety from a fan, by gluing together affine coordinate rings. Finally, we show how we can build the fan of a given toric variety. This chapter requires a lot of technical lemmas, but the effort is justified in Chapter 2, where we analyse the information of the variety that can be read directly from the fan.

Definition 1.0.5. A **torus** T of dimension r is an algebraic variety isomorphic to $(\mathbb{C}^*)^r$ where T inherits the group structure from the isomorphism.

Definition 1.0.6. A complex algebraic variety X is **toric** if there exists an embedding $\iota\colon (\mathbb{C}^*)^r \hookrightarrow X$, such that the image of ι is a dense open subset in the Zariski Topology and the usual multiplication in $T = \iota((\mathbb{C}^*)^r)$ extends to X (i.e. T acts on X). ι is sometimes called a **toric structure** on X.

Remark. Note that $\dim(X) = r$. We call $T = (\mathbb{C}^*)^r$ the **standard torus** of dimension r.

1.1 Lattices and fans.

Let $T \cong (\mathbb{C}^*)^r$ be a torus. A **one-parameter subgroup** is a group homomorphism $\lambda\colon \mathbb{C}^* \to T$. In particular, for the standard torus $(\mathbb{C}^*)^r$ all one-parameter subgroups are of the form (see [Hum75, §16]):

$$\lambda = \lambda^u(t) = (t^{u_1}, \dots, t^{u_r}), \qquad u = (u_1, \dots, u_r) \in \mathbb{Z}.$$

Therefore, the one parameter subgroups of a torus define a lattice $N \cong \mathbb{Z}^r$, called the **lattice of one-parameter subgroups**.

A **character** is a group homomorphism $\chi\colon T \to \mathbb{C}^*$. For the standard torus (see [Hum75, §16]) all characters are of the form

$$\chi = \chi^a(t_1, \dots, t_r) = t_1^{a_1} \cdot \dots \cdot t_r^{a_r}, \qquad a = (a_1, \dots, a_r) \in \mathbb{Z}^r.$$

The characters of a torus define a **character lattice** $M \cong \mathbb{Z}^r$ dual to N. There exists a natural pairing $\langle , \rangle\colon M \times N \to \mathbb{Z}$ defined in the following. First note that

$$\langle \chi^a, \lambda^u \rangle = \chi^a \circ \lambda^u\colon \mathbb{C}^* \longrightarrow \mathbb{C}^*,$$

is a group endomorphism of $\mathbb{C}^*$. Since all these endomorphisms have the form $t \to t^l$ for some $l \in \mathbb{Z}$ we can define

$$\langle \chi^a, \lambda^u \rangle = l \in \mathbb{Z}.$$

Finally, tensoring with $\mathbb{C}^*$ we get isomorphisms:

$$\begin{aligned} N \otimes \mathbb{C}^* &\cong T, & u \otimes z &\mapsto \lambda^u(z), \\ M \otimes \mathbb{C}^* &\cong \mathrm{Hom}(T, \mathbb{C}^*), & m \otimes z &\mapsto \chi^a(z). \end{aligned}$$

For this reason we will usually write T_N for the torus, to stress what the associated lattice is.

Example 1.1.1. $(\mathbb{C}^*)^r \hookrightarrow \mathbb{C}^r$ is a toric structure for $\mathbb{C}^r$ with $a \in N = \mathbb{Z}^r$, $b \in M = \mathbb{Z}^m$ and

$$\langle (a_1, \dots, a_r), (b_1, \dots, b_r) \rangle = \sum_{i=1}^{r} a_i b_i.$$

The following example is generalised in Chapter 5:

Example 1.1.2. The conifold $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$ is toric with torus

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \ : \ t_i \in \mathbb{C}^*\}.$$

Definitions 1.1.3. Let $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^r$, $M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \mathbb{R}^r$. We define

$$\sigma = \{r_1 v_1 + \dots + r_s v_s \in N_{\mathbb{R}} \ : \ r_i \geq 0\} \subseteq N_{\mathbb{R}},$$

a **convex polyhedral cone**, and we call $S = \{v_1, \dots, v_s\}$ a set of **generators** for σ. σ is **rational** if the generators are chosen to be in N.

The **dimension**, $\dim(\sigma)$, of the cone σ is the dimension of its relative interior as a topological space. The **dual of a cone** σ is

$$\sigma^\vee := \{u \in M_{\mathbb{R}} \ : \ \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}.$$

A face τ of σ is

$$\tau = \sigma \cap m^\perp = \{v \in \sigma \ : \ \langle m, v \rangle = 0\}, \qquad \text{for some } m \in \sigma^\vee.$$

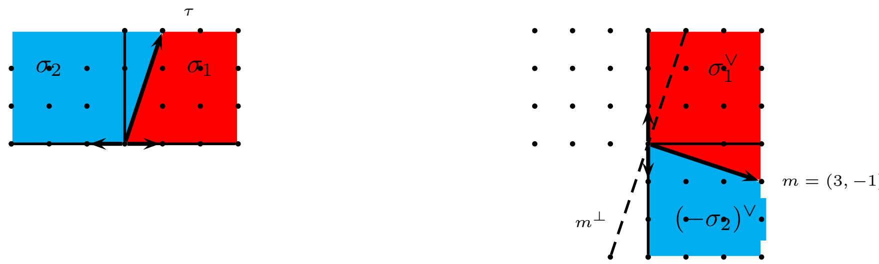
A face of codimension one is called a **facet** and a one-dimensional cone a **ray**.

If $\sigma \cap (-\sigma) = \{0\}$, then we say σ is **strongly convex**. If σ is also rational, then each ray ρ_v is spanned by vectors $v \in \sigma \cap N$, and v is called a **ray generator**. A cone is spanned by all its rayd generators, and we write $\sigma = \mathrm{Cone}(\rho_{v_1}, \dots, \rho_{v_r})$ or simply $\sigma = \mathrm{Cone}(v_1, \dots, v_r)$. From all generators of a ray ρ we can choose a **minimal** one, which generates $\rho \cap N$ as a semigroup. This will be useful later on.

Example 1.1.4. Consider Figure 1.1 (a) with lattice $N = \mathbb{Z}^2$ with the standard basis $\{e_1, e_2\}$ and $N_{\mathbb{R}} = \mathbb{R}^2$. We have six cones; two are of dimension 2 and may be expressed in terms of their minimal generators as:

$$\sigma_1 = \mathrm{Cone}(e_1, e_1 + 3e_2), \qquad \sigma_2 = \mathrm{Cone}(-e_1, e_1 + 3e_2).$$

The facets for σ_1 are (Cone(e_1) and $\tau = \mathrm{Cone}(-e_1, e_1 + 3e_2)$ and the facets for σ_2 are Cone($-e_1$) and τ. All five cones have as a face the trivial cone $(0, 0)$. The faces of σ_i are precisely the rays, since we are in dimension 2.



(a) Fan $\Sigma = \langle \sigma^1, \sigma^2 \rangle,\ m > 0$.

(b) Cones $\sigma_1^\vee$ and $(-\sigma_2)^\vee$, $m \geq 0$.

Figure 1.1: Fan Σ and its dual cones.

All the cones we use, unless otherwise stated, will be **strongly convex rational polyhedral cones (scrapcs)**, since they satisfy good properties that allow us to construct the desired varieties, as the following lemmas show.

Lemma 1.1.5 ([Ful93, section 1.2 (1), (2), (9), (4), (3)])**.** *For a convex polyhedral cone σ:*

(i) $(\sigma^\vee)^\vee = \sigma$.

(ii) If σ is scrapc, then so are its faces.

(iii) The dual of σ is a convex polyhedral cone.

(iv) A face of a face is a face, and the intersection of two faces is a face.

(v) If σ is a scrapc, then a face of $\sigma^\vee$ has the form $\sigma^\vee \cap \tau^\perp$, where τ is a face of σ.

Remark. Note that the cones σ_i in Example 1.1.4 are scrapc and so are their faces by part (ii) of this lemma.

Given rational generators for σ, say $v_1, \ldots, v_s$, we often need to find generators for $\sigma^\vee$. We assume that σ spans $N_\mathbb{R}$ as a vector space (otherwise we restrict to the subspace spanned by σ). Hence, for each facet τ we have a unique minimal vector $u_\tau \in \sigma^\vee$ such that $\langle u_\tau, v \rangle \geq 0,\ \forall v \in \sigma$ and $u_\tau \perp \tau$. Doing this for every facet we get generators for $\sigma^\vee \subset M_\mathbb{R}$ (see [Ful93, section 1.2 (8)] for details).

Lemma 1.1.6 (Gordan's lemma: [Ful93, section 1.2 Prop. 1])**.** *If σ is a rational convex polyhedral cone, then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup.*

Lemma 1.1.7 ([Ful93, section 1.2 (13)])**.** *If a cone is strongly convex, then its generators are given by its rays.*

Example 1.1.8. In Figure 1.1, from Example 1.1.4, $m = (3, -1)$ is the unique minimal vector u_τ of $\sigma_1^\vee$ such that $u_\tau \perp \tau$ or in other words, m is the inner pointing vector perpendicular to τ. Following the same procedure for e_1 we obtain the cone $\sigma_1^\vee$ as pictured in Figure 1.1 (b). Note that $\sigma_1^\vee$ is strongly convex. However, strong convexity is not generally preserved by duality, just convexity. Indeed $\tau^\vee = \mathrm{Cone}(m, -m, e_2)$ and $\tau^\vee = (-(\tau^\vee)) = \mathbb{R}\langle m \rangle \neq (0, 0)$. It is easy to check that $S_\sigma = \mathbb{Z}\langle e_2, m, e_1 \rangle$.

Some scrapcs will be of special interest in Chapter 2.

Definition 1.1.9. Let $\sigma \subseteq N_{\mathbb{R}}$ be a scrapc,

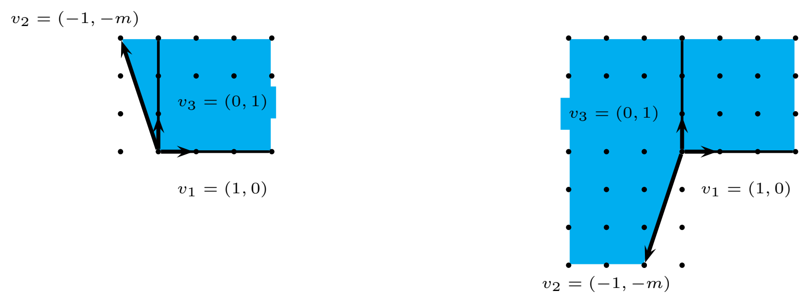
(i) σ is **smooth** if its minimal generators form part of a $\mathbb{Z}$-basis for N.

(ii) σ is **simplicial** if its minimal generators are linearly independent over $N_{\mathbb{R}}$.

Remarks 1.1.10. *These properties descend to faces, i.e. a face of a simplicial cone is simplicial and a face of a smooth cone is smooth. Moreover*

(i) If a cone σ is not smooth, then none of the cones containing σ are smooth.

(ii) Smooth cones are simplicial.



(a) $m < 0$.

(b) $m > 0$.

Figure 1.2: Fans of $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$.

Example 1.1.11. In Figure 1.1 (a) both cones are simplicial but not smooth. In Figure 1.2 both cones are smooth. For an example of a non-simplicial cone, we need to go to higher dimensions. For instance, the cone in Figure 5.1 is not simplicial, since it is generated by 4 rays.

Definition 1.1.12. A **fan** $\Sigma \subseteq N_{\mathbb{R}}$, is a finite collection of scrapcs in $N_{\mathbb{R}}$ satisfying:

- Each face of a cone in Σ is a cone in Σ.
- The intersection of two cones in Σ is a face of each of them.

The set of cones of dimension k is $\Sigma(k)$. The **support** of Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

We will write $\Sigma = \langle \sigma_i \rangle$ for a fan formed by cones σ_i and all their faces (providing they satisfy the definition).

There are obvious definitions for fans generalising the ones in Definition 1.1.9:

Definition 1.1.13. Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan,

(i) Σ is **smooth** if all its cones are smooth.

(ii) Σ is **simplicial** if all its cones are simplicial.

(iii) Σ is **complete** if $|\Sigma| = N_{\mathbb{R}}$.

Remark. We will assume that $\{v_\rho\}_{\rho\in\Sigma(1)}$ span $N_\mathbb{R}$ (i.e. $|\Sigma(1)| \geq \dim(N_\mathbb{R})$), since if this is not the case then we can always change N by a projection in some lattice of dimension n.

Lemma 1.1.14 ([Ful93, section 1.2 Prop. 2])**.** *Let $\tau = \sigma \cap m^\perp$, $m \in \sigma^\vee \cap M$ be a face of a cone σ then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m).$$

Lemma 1.1.15 (Separation lemma, [Ful93, section 1.2 (12)])**.** *If σ_1, σ_2 are convex polyhedral cones and $\tau = \sigma_1 \cap \sigma_2$ is a face of each, then for all m in the relative interior of $\sigma_1^\vee \cap (-\sigma_2)^\vee$,*

$$\tau = \sigma_1 \cap m^\perp = \sigma_2 \cap m^\perp. \tag{1.1}$$

Example 1.1.16. Figure 1.1 (a) shows the fan $\Sigma = \langle\sigma_1, \sigma_2\rangle$, where $\tau = \sigma_1 \cap \sigma_2$, as described in examples 1.1.4 and 1.1.8. In Figure 1.1 (b) we can see how Lemma 1.1.15 works. The dual cone of σ_1 (this is $\sigma_1^\vee$) and the negative of the dual of σ_2, i.e. $\sigma_2^\vee$ share the ray generated by m, which is the relative interior of $\sigma_1^\vee \cap (-\sigma_2)^v$, and m defines τ as in (1.1).

Note also that when $N = \mathbb{Z}^n$, as in this example, the algorithm described after Lemma 1.1.5 to find dual cones consists simply in finding the perpendicular inner vectors to each of the faces.

Lemma 1.1.17. *If $\tau = \sigma_1 \cap \sigma_2$ is a face of two cones, then $S_\tau = S_{\sigma_1} \cap S_{\sigma_2}$.*

Proof. It is a fact from cone theory that $\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee \subset \tau^\vee$. Therefore $S_{\sigma_1} + S_{\sigma_2} \subseteq S_\tau$.

Let $u \in S_\tau$ and $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ like in Lemma 1.1.15. Hence, Lemma 1.1.14 applied to σ_1 gives $u = v - km$ for $v \in S_{\sigma_1}, k \geq 0$. However $-m \in S_{\sigma_2}$ so $u \in S_{\sigma_1} + S_{\sigma_2}$. □

1.2 Building the toric variety of a fan by gluing affine coordinates.

Given a fan, we will construct an affine coordinate ring for each of the cones and then show how they glue together via common faces, using the properties in section 1.1.

Let $N \cong \mathbb{Z}^r$ be a lattice and $\sigma \subset N_\mathbb{R}$ a cone, such that $\dim(\sigma) = r$. For the semigroup S_σ in Lemma 1.1.6, we get a finitely generated commutative $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$. Indeed, by lemmas 1.1.14 and 1.1.15, σ is generated by minimal generators $v_1, \ldots, v_m$, so $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{v_1}, \ldots, \chi^{v_m}]$ where χ^{v_i} is the character corresponding to v_i. Moreover

$$\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[M] \tag{1.2}$$

where M is the dual of N. Hence $\mathbb{C}[S_\sigma]$ is an integral domain.

It is a well known construction in commutative algebra (see [AM69]) that choosing generators for $\mathbb{C}[S_\sigma]$, say $X_1, \ldots, X_m$, we get

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \ldots, X_m]\big/ I$$

where I is the ideal generated by the relations among those generators. This is the ring over the affine variety $U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

Taking the contravariant functor Spec in (1.2) we arrive at:

$$T_N = \operatorname{Spec}(\mathbb{C}[M]) \subseteq \operatorname{Spec}(\mathbb{C}[S_\sigma]) = X_\sigma.$$

Since $\dim \sigma = \dim N$ by assumption, the closure of T_N is X_σ. Therefore we have proved:

Proposition 1.2.1. *Given a scrapc $\sigma \subseteq N_{\mathbb{R}}$ of dimension r,*

$$X_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

is an affine toric variety.

Now that we have constructed affine varieties for cones, we want to use this idea to construct toric abstract algebraic varieties from fans in a consistent way. Suppose we have a fan $\Sigma \subseteq N_{\mathbb{R}}$ with a finite number of cones σ_i in it. By the construction we have just done, for each σ_i we have a toric affine variety U_{σ_i}, with coordinate ring $\mathbb{C}[S_{\sigma_i}]$. We want to glue them according to Definition 0.0.1. Since the number of cones is finite, it is enough to show the case of two cones σ_1 and $\sigma_2 \in \Sigma$ with a face in common $\tau = \sigma_1 \cap \sigma_2$.

Given $\sigma_1, \sigma_2 \in \Sigma$ with a face τ in common, by Lemma 1.1.14:

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_i} + \mathbb{Z}(-m)] \cong \mathbb{C}[S_{\sigma_i}]_{\chi^m}$$

where χ^m is the generator of the coordinate ring corresponding to m and $\mathbb{C}[S_{\sigma_i}]_{\chi^m}$ is the localisation at this element. Hence, by Lemma 1.1.15

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}. \tag{1.3}$$

We can define $U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_i}])$, which is an affine toric variety and (1.3) gives us an isomorphism:

$$g_{\sigma_2,\sigma_1} \colon (U_{\sigma_1})_{\chi^m} \longrightarrow (U_{\sigma_2})_{\chi^{-m}}$$

which satisfies the Definition 0.0.1 trivially, since it is the identity in $U_\tau \subset U_{\sigma_1}$ and $X_\Sigma := \bigsqcup U_\sigma \big/ \sim$.

Theorem 1.2.2. *Given a fan Σ in $N_{\mathbb{R}}$, X_Σ is toric.*

Proof. Since $\forall \sigma \in \Sigma$ σ is a scrapc, $\{0\}$ is a face of σ and $T_N = \operatorname{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^n \subset U_\sigma$, $\forall \sigma \in \Sigma$. We glue all these cones together, therefore $T_N \subseteq X_\Sigma$. The action of T_N on U_σ extends to X_Σ since g_{σ_2,σ_1} is the identity in $U_{\sigma_1} \cap U_{\sigma_2}$, and therefore it agrees on intersections. □

Example 1.2.3. Consider the fan in Figure 1.1. The coordinate rings for the dual cones are

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x_2, x_1^3 x_2^{-1}], \qquad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[x_1, x_1^{-3} x_2]$$

and the gluing map in the coordinate rings is given by:

$$g^*_{\sigma_2\sigma_1} \colon \mathbb{C}[x_2, x_1^3 x_2^{-1}]_{x_1^3 x_2^{-1}} \cong \mathbb{C}[x_1, x_1^{-3} x_2]_{x_1^{-3} x_2}$$

determining the gluing map between the affine varieties.

1.3 Building the fan of a toric variety.

Suppose we have a toric variety X with torus $T \subseteq X$, $T \cong (\mathbb{C}^*)^r$. To construct the fan consider the lattice given by one parameter subgroups, $N = \mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^r$.

A one-parameter subgroup $\widetilde{\Psi} \in N$ can be seen as $\Psi\colon \mathbb{C}^* \to X$ by $\Psi(t) := \widetilde{\Psi}(t) \cdot 1$ where $1 \in T$ is the identity of the torus. Since $\overline{T} = X$ we have that $\lim_{t\to 0} \Psi(t) \in X$ and $Z_\Psi := \overline{T \cdot \lim_{t\to 0} \Psi(t)}$ is a T-invariant subvariety of X.

As we will see more formally later on, all the T-invariant subvarieties of a variety X_Σ are in 1:1 correspondence with the cones in Σ. Therefore we need to include in the same cone those one-parameter subgroups which give the same toric subvariety. That is Ψ, Ψ' are in the same cone if, and only if, $Z_\Psi = Z_{\Psi'}$. This is an equivalent relation so we can split $N_\mathbb{R}$ into equivalent classes (cones). Note that we are using equality of subvarieties. If $Z_\Psi \subsetneq Z_{\Psi'}$, then Ψ and Ψ' are not in the same cone. As we will see later this will force the cone of Ψ' to be a face of the cone of Ψ.

Indeed, by the splitting into equivalent classes, for $\Psi \in N$ we have exactly one $\sigma \in \Sigma$ such that Ψ is in the relative interior of σ with $Z_\sigma = Z_\Psi$.

Example 1.3.1. We aim to show that the fans of $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ are the ones pictured in Figure 1.2. Let $p = ([z_0 : z_1], \theta) \in \mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ given in homogeneous coordinates. Note that $([0:0], \theta)$ can never be a point, (z_0, z_1) are related by a linear ratio, and θ can take local arbitrary values. We can cover the variety with the usual cover U, V for $\mathbb{CP}^1$ and take local coordinates for the open sets:

$$\begin{aligned} u &= \frac{z_1}{z_0}, & \zeta_U &= \frac{\theta}{z_0^m}, \quad \text{in } U = \{z_0 \neq 0\}, \\ v &= \frac{z_0}{z_1}, & \zeta_V &= \frac{\theta}{z_1^m}, \quad \text{in } V = \{z_1 \neq 1\}. \end{aligned} \tag{1.4}$$

The transition functions for $U \cap V$ are:

$$v = \frac{1}{u}, \qquad \zeta_V = u^{-m} \zeta_U \tag{1.5}$$

which are precisely the ones for $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$.

The embedding of the torus and the torus action are:

$$\begin{aligned} T = (\mathbb{C}^*)^2 &\hookrightarrow \mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m)) \\ (t_1, t_2) &\hookrightarrow ([1 : t_1], t_2^m) \end{aligned}$$

$$(t_1, t_2) \cdot ([x_0 : x_1], \theta) = ([x_0 : x_1 \cdot t_1], \theta\ t_2^m).$$

This seems like an arbitrary embedding, but it is the one which agrees in the intersection $U \cap V$ with the transition functions (1.4) and (1.5).

To find the fan we look for limits of all the possible one-parameter subgroups $\Psi_{a,b}(t) = ([1 : t^a], t^b) \sim ([t^{-a} : 1], t^{b-ma})$, and the closure of their orbits (this is where we use that the embedding is compatible with the transition functions).

For instance, if $a > 0, b = 0$, then

$$\lim_{t\to 0} \Psi(t) = \lim_{t\to 0} ([1 : t^a], 1) = ([1:0], 1), \quad \overline{T \cdot ([1:0], 1)} = \mathcal{O}_{[1:0]}(m).$$

If $b > ma, \ a < 0$, then:

$$\lim_{t\to 0}\Psi(t) = \lim_{t\to 0}([1 : t^a], t^b) = \lim_{t\to 0}([t^{-a} : 1], t^{b-ma}) = ([0 : 1], 0),$$

and the closure is

$$\overline{T\cdot([0 : 1], 0)} = T\cdot([0 : 1], 0),$$

where we use the transition functions in $U\cap V$ in (1.5). The case where $b < ma, \ b < 0$ is particularly interesting, as we obtain:

$$\lim_{t\to 0}\Psi(t) = \lim_{t\to 0}([1 : t^a], t^b) = \lim_{t\to 0}([t^{-\frac{b}{m}} : t^{a-\frac{b}{m}}], 1) = ([0 : 0], 1) \notin X_\Sigma.$$

This happens because $\text{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ is not compact, so not all limits are reached. The other cases are done similar and they are summarised in tables 1.1 and 1.2.

$(a, b) \in \mathbb{Z}^2$	$\lim_{t\to 0}\Psi(t)$	$\overline{T\cdot\lim_{t\to 0}\Psi(t)}$
$a, b > 0$	$([1 : 0], 0)$	$\{([1 : 0], 0)\}$
$a = 0, b > 0$	$([1 : 1], 0)$	$\mathbb{CP}^1$
$a > 0, b = 0$	$([1 : 0], 1)$	$\mathcal{O}_{[1:0]}(m)$
$a < 0, b > am$	$([0 : 1], 0)$	$\{([0 : 1], 0)\}$
$a < 0, b = am$	$([0 : 1], 1)$	$\mathcal{O}_{[0:1]}(m)$
$am > b, b < 0$	$([0 : 0], 1)$	$\emptyset$
$a = 0, b = 0$	$([1 : 1], 1)$	$\text{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$

Table 1.1: Computation of the fan of $\text{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m)), \ m > 0$.

Hence, representing the values for $m > 0$ in $\mathbb{R}^2$ we get the fan in Figure 1.2 (b). The data for $m < 0$ gives the fan in Figure 1.2 (a). In example 3.2.8 we will recover the variety from its fan in homogeneous coordinates.

$(a, b) \in \mathbb{Z}^2$	$\lim_{t\to 0}\Psi(t)$	$\overline{T\cdot\lim_{t\to 0}\Psi(t)}$
$a, b > 0$	$([1 : 0], 0)$	$\{([1 : 0], 0)\}$
$a = 0, b > 0$	$([1 : 1], 0)$	$\mathbb{CP}^1$
$a > 0, b = 0$	$([1 : 0], 1)$	$\mathcal{O}_{[1:0]}(m)$
$a < 0, b > -am$	$([0 : 1], 0)$	$\{([0 : 1], 0)\}$
$a < 0, b = -am$	$([0 : 1], 1)$	$\mathcal{O}_{[0:1]}(m)$
$am > b$	$([0 : 0], 1)$	$\emptyset$
$a = 0, b = 0$	$([1 : 1], 1)$	$\text{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$

Table 1.2: Computation of the fan of $\text{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m)), \ m < 0$.

1.4 Morphisms

Definition 1.4.1. Let N_1, N_2 be two lattices with fans $\Sigma_1 \subseteq (N_1)_\mathbb{R}$ and $\Sigma_2 \subseteq (N_2)_\mathbb{R}$. A $\mathbb{Z}$-linear mapping $\overline{\phi}\colon N_1 \to N_2$ with induced map

$$\overline{\phi}_\mathbb{R}(z\otimes r) = \overline{\phi}(z)\otimes r$$

is **compatible** with Σ_1, Σ_2 if $\forall \sigma_1 \subseteq \Sigma_1$, $\exists \sigma_2 \subseteq \Sigma_2$ such that $\phi(\sigma_1) \subseteq \sigma_2$.

Definition 1.4.2. A toric morphism ϕ of normal varieties $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ is **toric** if when restricted to the torus $T_{N_1} \subseteq X_{\Sigma_1}$, then $\phi|_{T_{N_1}}\colon T_{N_1} \to T_{N_2}$ is a group homomorphism.

The purpose of this section is to show that for normal toric varieties, there is a one-to-one relation between maps compatible with the fan structures and toric morphisms.

Lemma 1.4.3. *Given two affine toric varieties $V_i = \operatorname{Spec}(\mathbb{C}[S_{\sigma_i}])$ with $\sigma_i \subseteq (N_i)_{\mathbb{R}}$, $i = 1, 2$ a morphism $\phi\colon V_1 \to V_2$ is toric if and only if the corresponding map of coordinate rings $\phi^*\colon \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $\hat{\phi}\colon S_2 \to S_1$.*

Proof. Suppose $\phi^*\colon \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $\hat{\phi}\colon S_2 \to S_1$. Since the character lattice of T_{N_i} is $M_i = \mathbb{Z}S_i$ we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[S_2] & \xrightarrow{\phi^*} & \mathbb{C}[S_1] \\ \downarrow & & \downarrow \\ \mathbb{C}M_2 & \longrightarrow & \mathbb{C}[M_1]. \end{array}$$

Since $T_{N_i} = \operatorname{Hom}_{\mathbb{Z}}(M_i, \mathbb{C}^*)$, taking Spec, we have that $\phi|_{T_{N_1}}$ is a group homomorphism and therefore ϕ is toric.

Conversely, a toric morphism induces a diagram as above, and by restriction to lattice generators we get a group homomorphism $\hat{\phi}\colon S_2 \to S_1$ inducing ϕ^*. ☐

Lemma 1.4.4. *Let X_1, X_2 be toric varieties. Any toric morphism $\phi\colon X_1 \to X_2$ is equivariant, i.e. the following diagram*

$$\begin{array}{ccc} T_{N_1} \times X_1 & \xrightarrow{\Phi_1} & X_1 \\ {\scriptstyle \phi|_{T_{N_1}} \times \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ T_{N_2} \times X_2 & \xrightarrow{\Phi_2} & X_2 \end{array} \tag{1.6}$$

commutes, where Φ_i is the action of T_{N_i} on X_i.

Proof. If we restrict X_i to T_{N_i} in (1.6), the diagram certainly commutes, since ϕ_i and Φ_i are group homomorphisms, and since T_{N_i} is Zariski dense in X_i the whole diagram in (1.6) commutes. ☐

The following theorem is crucial in proving the correspondence between the concepts of compatible maps and toric morphisms.

Theorem 1.4.5. *Suppose we have a scrapc $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ and a lattice morphism $\overline{\phi}\colon N_1 \to N_2$. The induced homomorphism of tori*

$$\phi\colon T_{N_1} \to T_{N_2}; \qquad x \otimes z \longmapsto \overline{\phi}(x) \otimes z$$

extends to a map of affine toric varieties $\phi\colon V_{\sigma_1} \to V_{\sigma_2}$ if and only if $\overline{\phi}_{\mathbb{R}}$ is compatible with σ_1 and σ_2.

Proof. Recall how $\overline{\phi}\colon N_1 \to N_2$ induces all other maps:

$$\begin{aligned} &\phi|_{T_{N_1}}\colon T_{N_1} \to T_{N_2}; && z \otimes_{\mathbb{Z}} t \longmapsto \overline{\phi}(z) \otimes_{\mathbb{Z}} t \quad \text{for } z \in N_1, t \in \mathbb{C}^*, \\ &\phi_{\mathbb{R}}\colon (T_{N_1})_{\mathbb{R}} \to (T_{N_2})_{\mathbb{R}}; && z \otimes_{\mathbb{Z}} t \longmapsto \overline{\phi}(z) \otimes_{\mathbb{Z}} t \quad \text{for } z \in N_1, r \in \mathbb{R}. \end{aligned}$$

We denote $\phi|_{T_{N_1}}$ for the map on the torus and $\phi|_{U_{\sigma_1}}$ for the map on the variety.

Suppose $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. Let $p \in U_{\sigma_1} \setminus T_{N_1}$. Then, for some $\Psi \in \sigma_1 \cap N_1$, $s \in T_{N_1}$ we have $p = \lim_{t\to 0} \Psi(t) \cdot s$. Define $\phi\colon U_{\sigma_1} \to U_{\sigma_2}$ as

$$\phi|_{U_{\sigma_1}}(p) := \phi|_{U_{\sigma_1}}(\lim_{t\to 0} \Psi(t) \cdot s) = \lim_{t\to 0}(\overline{\phi} \circ \Psi(t)) \cdot \phi|_{T_{N_1}}(s) \tag{1.7}$$

for $p \in U_{\sigma_1} \setminus T_{N_1}$ and $\phi(p) \in U_{\sigma_2}$ since $\overline{\phi_{\mathbb{R}}}(\sigma_1) \subseteq \sigma_2$.

When $p \in T_{N_1}$, p is necessarily $\Psi(t) \cdot s$ for $s \in T_{N_1}$ and $\psi = 0 \in \sigma_1 \cap N$, in which case (1.7) restricts to $\phi|_{T_{N_1}}$, so we have extended $\phi|_{T_{N_1}}$ to $\phi U_{\sigma_1} \to U_{\sigma_2}$.

Conversely, if $\phi|_{T_{N_1}}$ extends to $\phi|_{U_{\sigma_1}}$ then for $p \in U_{\sigma_1} \setminus T_{N_1}$ as before, we get the expression in (1.7). If $\phi|_{U_{\sigma_1}}(p) \in U_{\sigma_2}$, then $\lim_{t\to 0}(\overline{\phi} \circ \psi(t))$ must reach its limit in U_{σ_2} but this requires that $\overline{\phi} \circ \psi \in \sigma_2 \cap N$, for all $\psi \in \sigma_1 \cap N$, i.e. $\overline{\phi}(\psi) \in \sigma_2 \cap N$ and $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. □

Theorem 1.4.6. *Let $\Sigma_i \subseteq (N_i)_{\mathbb{R}}$, $i = 1, 2$ be fans. If $\overline{\phi}\colon N_1 \to N_2$ is a $\mathbb{Z}$-linear map compatible with Σ_1, Σ_2, then there exists a toric morphism $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ such that*

$$\phi|_{T_{N_1}} = \overline{\phi} \otimes 1\colon N_1 \otimes \mathbb{C}^* \to N_2 \otimes \mathbb{C}^*.$$

Conversely, given a toric morphism $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ it induces a $\mathbb{Z}$-linear map $\overline{\phi}\colon N_1 \to N_2$ compatible with Σ_1 and Σ_2.

The proof of Theorem 1.4.6 requires the Orbit-Cone Correspondence (Theorem 2.3.5), so we delay its proof until section 2.3, but we can prove one of the two implications now:

Theorem 1.4.7. *Let N_1, N_2 be lattices and let $\Sigma_1 \subseteq (N_1)_{\mathbb{R}}$, $\Sigma_2 \subseteq (N_2)_{\mathbb{R}}$ be fans with $\overline{\phi}\colon N_1 \to N_2$ a $\mathbb{Z}$-linear morphism compatible with Σ_1, Σ_2, then the $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2}$ induced by $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2 \in \Sigma_2$ glue to form a morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$. Moreover this map is toric.*

Proof. We take the open cover $\{U_{\sigma_i}\}_{\sigma_i \in \Sigma_1}$ of X_{Σ_1} and Theorem 1.4.5 gives us induced morphisms of affine varieties:

$$\phi_{\sigma_i} : U_{\sigma_i} \to U_{\overline{\phi}(\sigma_i)} \qquad x \otimes z \longmapsto \overline{\phi}(x) \otimes z.$$

Now, let $\sigma_1, \sigma_1' \in (\Sigma_1)_{\mathbb{R}}$ be such that $\sigma_1 \cap \sigma_1' = \tau \in (\Sigma_1)_{\mathbb{R}}$ is a maximal face of both. By the proof of Proposition 1.2.1 and as a consequence of Lemma 1.1.17, we obtain $U_{\sigma_1} \cap U_{\sigma_1'} = U_{\sigma_1 \cap \sigma_1'} = U_\tau$. Hence, for $x \in \tau \cap N_1$

$$\phi_{\sigma_1}|_{U_{\sigma_1} \cap U_{\sigma_1'}}(x \otimes z) = \overline{\phi}(x) \otimes z = \phi_{\sigma_1'}|_{U_{\sigma_1'} \cap U_{\sigma_1}}(x \otimes z)$$

using again Lemma 1.1.17. If we take $\sigma_1 = \{0\}$ then $U_{\{0\}} = T_{N_1}$ and we get $\phi_{\{0\}} : T_{N_1} \to T_{N_2}$ which is a group homomorphism by definition, so $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism. □

Chapter 2

Applications of the fan construction

2.1 Correspondence between fans and normal toric varieties

We have described algorithms to pass from toric varieties to fans and vice versa. However it is interesting to know whether this is always possible, i.e. in which case there is a bijection between fans and toric varieties. We recall what a normal variety is:

Definition 2.1.1. A ring R with field of fractions K is normal if every element of K which is integral over R (i.e. a root of monic polynomial in $R[x]$) is actually in R. A variety X is **normal** if it is irreducible and the local rings $\mathcal{O}_{X,p}$ are normal for all $p \in X$.

Proposition 2.1.2. *If a toric variety X whose torus is a proper subset is normal, then we can associate to it a fan Σ , so that $X \cong X_\Sigma$. Conversely from any given fan Σ we can construct a toric variety X_Σ which is normal.*

Proof (Sketch). Normality is a local property, therefore X_Σ is normal if and only if U_σ is normal $\forall \sigma \in \Sigma$. Let $V = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = U_\sigma$, for $S_\sigma = \sigma^\vee \cap M$, $\sigma \subset N_\mathbb{R}$ a scrapc with the same dimension as the lattice (the case of a smaller dimension is similar, but with torus factors). Let $\rho_{v_1}, \dots, \rho_{v_r}$ be the rays of σ, i.e. $\sigma = \mathrm{Cone}(v_1, \dots, v_r)$. Then $\sigma^\vee = \bigcap_{i=1}^r \rho_i^\vee$, and intersecting with M:

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\bigcap_{i=1}^r S_{\rho_i}] = \bigcap_{i=1}^r \mathbb{C}[S_{\rho_i}].$$

Since the intersection of normal rings is normal, it is enough to show $\mathbb{C}[S_\rho]$ is normal for any ray $\rho \in \sigma(1)$. Let $v = v_\rho$, then since v is minimal, we can take a basis of N such that

$$\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}] = \mathbb{C}[x_1, \dots, x_n]_{x_2 \cdots x_n}$$

is normal since it is a localisation of the normal ring $\mathbb{C}[x_1, \dots, x_n]$.

The proof that a normal toric variety arises from a fan is a consequence of Sumihiro's work [Sum75, Sum74]. A more modern version can be found in [Fin89, Fin93] □

To agree with our convention that all varieties will be separated, (Definition 0.0.1), we need:

Proposition 2.1.3 ([CLSar, Theorem 3.1.5]). *Given a fan Σ, the variety X_Σ is separated.*

Notation. Since we are interested in analysing geometric properties arising from the combinatorics of the fan we will restrict ourselves only to normal toric varieties and we will often just call them toric varieties, implying they are normal. We will write X_Σ to denote a normal toric variety arising from a fan Σ.

Corollary 2.1.4. *Let $\Sigma \subseteq N_\mathbb{R}$ be a fan for some lattice N, with $n = \dim N = \dim \Sigma$. Let $\overline{f} \in SL(N, \mathbb{Z})$ and $\widetilde{\Sigma} = \overline{f}(\Sigma)$. Then $X_\Sigma \cong X_{\widetilde{\Sigma}}$.*

Proof. Let $\overline{g} \in SL(N, \mathbb{Z})$ such that $\overline{f} \circ \overline{g} = \overline{g} \circ \overline{f} = \mathrm{Id}_N$. In fact, all the invertible maps compatible with Σ which preserve the lattice are of these form. Let $\widetilde{\Sigma} = \overline{f}(\Sigma)$. There exist induced functions $f \colon X_\Sigma \to X_{\widetilde{\Sigma}}$, $g \colon X_{\widetilde{\Sigma}} \to X_\Sigma$, with $g \circ f = \mathrm{Id}_{X_\Sigma}$, $f \circ g = \mathrm{Id}_{X_{\widetilde{\Sigma}}}$, by theorem 1.4.6 so $X \cong X_{\widetilde{\Sigma}}$. □

We can summarise the pairing between fans and normal varieties of Proposition 2.1.2 and Corollary 2.1.4 in the following

Corollary 2.1.5. *There is a one-to-one correspondence (up to isomorphism) between fans and normal toric varieties.*

2.2 Smoothness and compactness

Recall that a cone $\sigma \subset N_\mathbb{R}$ is smooth if its primitive generators are part of a basis of the lattice N.

Proposition 2.2.1. *A toric variety X_Σ is smooth if and only if Σ is smooth.*

Proof. First we reduce to the affine case. Note that Σ is smooth if and only if all its cones are smooth. Also a variety is smooth if and only if it is smooth in all open sets of an open cover. Therefore it suffices to show it in the affine case, i.e.:

$$\sigma \text{ is smooth} \Leftrightarrow U_\sigma \text{ is smooth.}$$

Also note that if an affine variety is smooth, then it is normal and by Corollary 2.1.5, of the form U_σ, i.e. arising from a scrapc σ. Suppose we start with a smooth scrapc σ:

$$\sigma = \mathrm{Cone}(e_1, \ldots, e_r) \subset \mathbb{R}\langle e_1, \ldots, e_r, e_{r+1}^{\pm 1}, \ldots, e_n^{\pm 1}\rangle.$$

Then, its dual cone is $\sigma^\vee = \mathrm{Cone}(e_1, \ldots, e_r, e_{r+1}^{\pm 1}, \ldots, e_n^{\pm 1})$ and the ring of coordinates is:

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap \ M] \cong \mathbb{C}[e_1, \ldots, e_r, e_{r+1}^{\pm 1}, \ldots, e_n^{\pm 1}],$$

giving the affine variety $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) \cong \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$ which is smooth.

Conversely, suppose we have $\sigma \subseteq N_\mathbb{R}$ such that U_σ is smooth and let $n = \dim U_\sigma = \dim N_\mathbb{R}$. We can reduce to the case where $\dim \sigma = r = n$, since in any other case $U_\sigma \cong U_{\widetilde{\sigma}} \times (\mathbb{C}^*)^{n-r}$, where $\widetilde{\sigma} \subseteq N/\sigma^\perp = \widetilde{N}$ with $\dim \widetilde{\sigma} = \dim \widetilde{N}$. In this case U_σ is smooth if and only if $U_{\widetilde{\sigma}}$ is smooth and σ is smooth if and only if $\widetilde{\sigma}$ is smooth. Therefore we suppose $r = n$.

By [CLSar, lemma 1.3.10] the minimum number of generators for S_σ, is $\dim T_{p_\sigma}$, so:

$$n = \dim U_\sigma = \dim T_{p_\sigma} U_\sigma \geq \{\text{edges } \rho \subseteq \sigma^\vee\} = \dim \sigma \geq n.$$

Hence, $\sigma^\vee$ has n edges, generating $M \subseteq \mathbb{Z}^n$ and by Lemma 1.1.5 $\sigma = (\sigma^\vee)^\vee$ is smooth, since duality preserves smoothness. □

Example 2.2.2. In the fans in Figure 1.2 we can see that all the two top-dimensional cones are generated (as a semigroup) by the rays surrounding them and therefore their toric varieties are smooth. To see the difference, consider the fan in Figure 2.1 for $\mathbb{C}^2/\mathbb{Z}_2$. We know it has a singularity at the origin. This is reflected in the fan. Rather than 2 vectors to generate the 2 dimensional cone as a semigroup, we need 3. This results in a 3-dimensional affine space ($\mathbb{C}^3$) cut out by a monomial ideal ($\langle xy = z^2 \rangle$) which is the reason for the singularity.

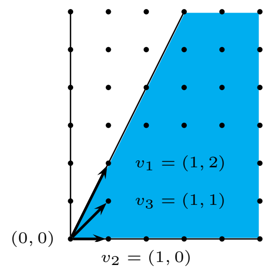


Figure 2.1: Fan of $\mathbb{C}^2/\mathbb{Z}_2$.

Proposition 2.2.3. *A toric variety X_Σ is compact if and only if Σ is complete.*

Proof. Suppose $\bigcup_{\sigma\in\Sigma} \sigma \subsetneq N_\mathbb{R}$, then $\exists \Psi\colon \mathbb{C}^* \longrightarrow X$ a one-parameter subgroup such that its limit as $t \to 0$ is not in X so X_Σ is not compact.

Conversely, if all one-parameter subgroups converge as $t \to 0$ then X is compact. □

Example 2.2.4. Just by looking at its fan in Figure 1.2, we know that the variety $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ is not compact. This is not surprising, since the total space of a vector bundle over a variety is never compact.

2.3 The Orbit-Cone Correspondence

Given the cone $\sigma \subseteq \Sigma \subseteq N_\mathbb{R}$ and its affine variety U_σ, all the geometric information (subvarieties, one-parameter subgroups, singularities) invariant under the torus action is contained in the combinatorics of the cone. In this section we consider a fixed $\sigma \subseteq \Sigma \subseteq N_\mathbb{R}$.

The following lemma extends to semigroup homomorphisms the bijection between points of an affine variety and maximal ideals of its ring of regular functions:

Lemma 2.3.1 ([CLSar, Prop. 1.3.1])**.** *Let* $\mathbf{V} = \mathrm{Spec}(\mathbb{C}[S])$ *be an affine toric variety (not necessarily normal) where S is a semigroup with basis $m_1, \dots, m_r$. Then, there is a bijection:*

$$\begin{aligned} U &\Longleftrightarrow \mathrm{Hom}(S, \mathbb{C}) \\ p &\longmapsto (\delta\colon m \mapsto \chi^m(p)) \\ (\delta(m_1), \dots, \delta(m_r)) &\longleftarrow (\delta\colon S \to \mathbb{C}). \end{aligned} \tag{2.1}$$

We have a semigroup homomorphism $S_\sigma \longrightarrow \mathbb{Z}_2 \subset \mathbb{C}^* \subset \mathbb{C}$ given by:

$$m \longmapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

By equation (2.1) in Lemma 2.3.1, (2.2) gives a unique **distinguished point** γ_σ for σ.

Proposition 2.3.2. *Let $\sigma \subset N_\mathbb{R}$, $u \in N$, then:*

$$u \in \sigma \Leftrightarrow \lim_{t\to 0} \lambda^u(t) = \gamma_\sigma.$$

Proof.

$$\begin{aligned} \lim_{t\to 0} \lambda^u(t) \text{ exists in } U_\sigma &\Leftrightarrow \lim_{t\to 0} \chi^m(\lambda^u(t)) \in \mathbb{C}\ \forall m \in S_\sigma \\ &\Leftrightarrow \langle m, u\rangle \geq 0\ \forall m \in \sigma^\vee \cap M \\ &\Leftrightarrow u \in (\sigma^\vee)^\vee = \sigma, \end{aligned}$$

where in the last equality we use Lemma 1.1.5.

If $u \in \sigma \cap N$, then $\lim_{t\to 0} \lambda^u(t)$ corresponds to the homomorphism

$$m \in \sigma^\vee \cap M \longrightarrow \lim_{t\to 0} t^{\langle m,u\rangle}.$$

by (2.1). If $u \in \text{RelInt}(\sigma)$ (the relative interior of σ with respect to the subspace topology), then $\langle m, u\rangle > 0$ for $m \in S_\sigma \cap \sigma^\perp$ and 0 otherwise, which is precisely γ_σ as in (2.2). ☐

Definition 2.3.3. The **torus orbit** of a cone $\sigma \subseteq \Sigma \subseteq N_\mathbb{R}$ is

$$O(\sigma) = T_N \cdot \gamma_\sigma \subset X_\Sigma.$$

Recall that if $\dim(\sigma) = \dim N_\mathbb{R}$, then γ_σ is fixed by T_N. Therefore it is natural to expect a dimension reversing correspondence between orbits and cones. The following way of expressing the orbits will be useful to prove the Orbit-Cone Correspondence:

Lemma 2.3.4 ([CLSar, Lem. 3.2.5])**.**

$$O(\sigma) = \{\gamma\colon S_\sigma \to \mathbb{C} \ :\ \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\} \cong \text{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).$$

Theorem 2.3.5 (Orbit-Cone Correspondence)**.** *Let X_Σ be the toric variety of $\Sigma \subseteq N_\mathbb{R}$, $\dim N_\mathbb{R} = n$.*

(i) There is a bijection

$$\begin{aligned} \{\sigma \subseteq \Sigma\} &\overset{1:1}{\longleftrightarrow} \{T_N\text{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \cong \text{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \end{aligned}$$

with $\dim O(\sigma) = n - \dim \sigma$.

(ii) The affine variety of a cone is the union of the orbits of its faces, i.e.:

$$U_\sigma = \bigcup_{\tau \text{ face of } \sigma} O(\tau).$$

(iii) τ *is a face of* σ *if and only if* $O(\sigma) \subseteq \overline{O(\tau)}$ *and*

$$\overline{O(\tau)} = \bigcup_{\tau \text{ is a face of } \sigma} O(\sigma)$$

where $\overline{O(\tau)}$ *is the closure of* $O(\tau)$ both *in the Zariski and classical topologies.*

Proof. We prove only parts (i) and (ii) although the statement of the dimension of σ in (i) will be proved later in Theorem 3.3.1 for a different version of this theorem for the closures of the orbits. A proof of (iii) can be found in [CLSar, Theorem 3.2.6].

Let O be a T_N-orbit. We know that X_Σ is covered by U_σ, and that these are T_N-invariant. So $O \subseteq U_\sigma$ for some σ. Since $U_{\sigma_1} = U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$ by construction, we can choose the smaller σ such that $O \subseteq U_\sigma$. Now let $p \in O$, then $p = s \cdot \lim_{t\to 0} \lambda^u(t)$, for $s \in (\mathbb{C}^*)^n$, $u \in \sigma$. Denote by δ the semigroup homomorphism corresponding to p under (2.1), and consider $m \in S_\sigma$ such that $\delta(m) \neq 0$. Then, since $p = \lim_{t\to 0} \lambda^u(t) \cdot s$ for some $s \in T_N$,

$$0 \neq \delta(m) = \chi^m(p) = \chi^m\left(\lim_{t\to 0} \lambda^u(t) \cdot s\right),$$

if and only if

$$\chi^m\left(\lim_{t\to 0} \lambda^u(t)\right) = \lim_{t\to 0} t^{\langle m,u\rangle} \neq 0,$$

if and only if $\langle m, u\rangle = 0$. i.e. if $\delta(m) \neq 0$, then m is in a face of $\sigma^\vee$. By Lemma 1.1.5, there is a face τ of σ such that:

$$\{m \in S_\sigma \ : \ \gamma(m) \neq 0\} = \sigma^\vee \cap \tau^\perp \cap M.$$

Hence $p \in U_\tau$ and hence $\tau = \sigma$ since σ was minimal. Now, by Lemma 2.3.4, $\gamma \in O(\sigma)$, so $O(\sigma) = O$ since they have a non-empty intersection (and orbits are either disjoint or equal).

To prove (ii), if τ is a face of σ, then $O(\tau) \subseteq U_\tau \subseteq U_\sigma$. By part (i) all orbits are of the form $O(\tau)$ and since U_σ is a union of orbits we are done. □

We immediately obtain the following

Corollary 2.3.6. $\overline{O(\sigma)}$*, for* $\sigma \in \Sigma$*, are all the* T_N *invariant subvarieties of* X_Σ*. In particular,* $D_\rho = \overline{O(\rho)}$*, for* $\rho \in \Sigma(1)$*, are all* T_N*-invariant codimension-*1 *subvarieties.*

Example 2.3.7. In tables 1.1 and 1.2 we presented the closure of all the T_N-orbits of $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ for any m for the different cones. Since we did this for all the cones of the fan Σ, these are all the orbits of $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$.

Observation 2.3.8. *By the Orbit Cone Correspondence (Theorem 2.3.5), the irreducible components of* $X_\Sigma \setminus T_N$ *are* $D_\rho = \overline{\rho}$ *for* $\rho \in \Sigma(1)$*, since* T_N *is an open dense set of* X_Σ *so its complement is a union of irreducible components and* $D_\rho \cap T_N = \emptyset$*.*

Now that we have the Orbit-Cone Correspondence we can finish the proof of Theorem 1.4.6. This shows the potential of Theorem 2.3.5 to determine the geometry of the toric variety from its fan.

Theorem (Theorem 1.4.6)**.** *Let* $\Sigma_i \subseteq (N_i)_\mathbb{R}$*,* $i = 1, 2$ *be fans. If* $\overline{\phi}\colon N_1 \to N_2$ *is a* $\mathbb{Z}$*-linear map compatible with* Σ_1, Σ_2*, then there exists a toric morphism* $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ *such that*

$$\phi|_{T_{N_1}} = \overline{\phi} \otimes 1\colon N_1 \otimes \mathbb{C}^* \to N_2 \otimes \mathbb{C}^*.$$

Conversely, given a toric morphism $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ *it induces a* $\mathbb{Z}$*-linear map* $\overline{\phi}\colon N_1 \to N_2$ *compatible with* Σ_1 *and* Σ_2*.*

Proof. The first assertion was proved in Theorem 1.4.7. For the second, let $u \in N_1$ induce the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T_{N_2}$. Since $\phi|_{T_{N_1}}$ is a group homomorphism we obtain by composition $\phi|_{T_{N_1}} \circ \lambda^u : \mathbb{C}^* \to T_{N_2}$ an element $\overline{\phi}(u) \in N_2$ and linearity is preserved.We have therefore defined $\overline{\phi} : N_1 \to N_2$ mapping Σ_1 into Σ_2. Now, we need to see that $\overline{\phi}$ is compatible with Σ_1 and Σ_2. Lemma 1.4.4 guarantees that T_{N_1}-orbits $O_1 \subseteq X_{\Sigma_1}$ are mapped to T_{N_2}-orbits and the Orbit-Cone Correspondence guarantees $O_i = O(\sigma_i)$ for some $\sigma_i \subset (\Sigma_i)_{\mathbb{R}}$, $i = 1, 2$. So, we have $\overline{\phi}(\sigma_1) \subseteq \sigma_2 \in \Sigma_2$, for some σ_2.

Let τ_1 be a face of σ_1. We have $\tau_2 \in \Sigma_2$ such that $\overline{\phi}(\tau_1) = \tau_2$ by the same argument. By Theorem 2.3.5 (iii), $O(\sigma_1) \subseteq \overline{O(\tau_1)}$ and since ϕ is continuous, $\phi(\overline{O(\sigma_1)}) \subseteq \overline{O(\tau_2)}$. So $O(\sigma_2) \subseteq O(\tau_2)$ and τ_2 is face of σ_2. We obtain that ϕ maps U_{ϕ_1} into U_{ϕ_2} and Theorem 1.4.5 tells us that $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$ so ϕ is compatible with Σ_1 and Σ_2. □

2.4 Computing divisor classes

Computing divisor classes can be a difficult problem in Algebraic Geometry. The following theorem is the basic tool to work with divisor classes:

Theorem 2.4.1 ([Har77, 6, Proposition 6.5 (c), p. 133])**.** *Let* U *be a nonempty Zariski open subset of a normal variety* X *and let* $D_1, \ldots, D_s$ *be the irreducible components of* $X \setminus U$ *that are prime divisors. Then the sequence*

$$\bigoplus_{j=1}^{s} \mathbb{Z}D_j \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \to 0$$

is exact, where the first map sends $\sum_{j=1}^{s} a_j D_j$ *to its divisor class in* $\mathrm{Cl}(X)$ *and the second is induced by restriction to* U*.*

However, in the case of toric varieties this result can be improved, as we will see in Theorem 2.4.4. This theorem will be crucial in Chapter 3. Furthermore, in example 2.4.7 we give two ways to compute divisor classes.

Definition 2.4.2. The group of torus-invariant Weil divisors of X_Σ is

$$\mathrm{Div}_{T_N}(X_\Sigma) := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \cong \mathbb{Z}^{\Sigma(1)} \subseteq \mathrm{Div}(X_\Sigma).$$

Since for all $D_\rho \in \mathrm{Div}_{T_N}(X_\Sigma)$, O_{X_Σ, D_ρ} is a discrete valuation ring (DVR) (see [Har77, p. 130]), it has an associated valuation (see [AM69, 1, Prop. 9.2, p. 94]):

$$\nu_\rho := \nu_{D_\rho}\colon \mathbb{C}(X_\Sigma)^* \longrightarrow \mathbb{Z}.$$

Note that for $m \in M$, $\chi^m\colon T_N \to \mathbb{C}^*$ extends to $X_\Sigma \dashrightarrow \mathbb{C}^*$ since T_N is Zariski open.

Proposition 2.4.3. *Let* X_Σ *be a toric variety,* $\rho \in \Sigma(1)$ *with minimal generator* u_ρ*, then*

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle. \tag{2.3}$$

Moreover the principal divisor of χ^m *is*

$$\operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho. \tag{2.4}$$

Proof. First we choose a convenient basis for N. Since U_ρ is minimal we can complete a basis $\{e_1, \dots, e_n\}$ of N with $e_1 = u_\rho$ such that $N = \mathbb{Z}^n$ and $u_\rho = \operatorname{Cone}(e_1)$ and

$$U_\rho \cong \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_2^{\pm 1}, \dots, x_n^{\pm 1}]) = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n])_{x_1} = \mathbb{C} \times (\mathbb{C}^*)^{n-1},$$

where $x_1, \dots, x_n$ are the characters of the dual basis of $\{e_1, \dots, e_n\}$. Hence

$$\mathcal{O}_{X_\Sigma, D_\rho} \cong \mathbb{C}[x_1, \dots, x_n]_{x_1}.$$

By [GH78, pp. 130-131], the valuation for D_ρ is $\nu_\rho(f) = n \in \mathbb{Z}$ where

$$f \in \mathbb{C}(x_1, \dots, x_n)^*, \qquad f = x_1^n \frac{g}{h},$$

with g, h not divisible by x_1. Now, since $\chi^m \in \mathbb{C}(x_1, \dots, x_n)^*$ $\forall m \in M$

$$\nu_\rho(\chi^m) = \nu_\rho(x_1^{\langle m, e_1\rangle} \cdots x_n^{\langle m, e_n\rangle}) = \langle m, e_1\rangle = \langle m, u_\rho\rangle.$$

□

Theorem 2.4.4. *For a toric variety* X_Σ *without torus factors the sequence*

$$0 \longrightarrow M \longrightarrow \operatorname{Div}_{T_N}(X_\Sigma) \longrightarrow \operatorname{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.5}$$

is exact, where the first map sends m *to* $\operatorname{div}(\chi^m)$ *and the second map sends a divisor to its divisor class.*

Proof. From Theorem 2.4.1 and Observation 2.3.8 we have the exact sequence:

$$\operatorname{Div}_{T_N}(X_\Sigma) \longrightarrow \operatorname{Cl}(X_\Sigma) \longrightarrow \operatorname{Cl}(T_N) \longrightarrow 0.$$

Since $\mathbb{C}[X_1, \dots, X_n]$ is a unique factorization domain (UFD) (see [GH78, p. 10]), so is

$$\mathbb{C}[X_1, \dots, X_n]_{X_1 \cdots X_n} \cong \mathcal{O}_{(\mathbb{C}^*)^n} \cong \mathcal{O}_{T_N},$$

thus its class group vanishes, $\operatorname{Cl}(T_N) = 0$ (see [Har77, 6, Prop 6.2, p. 131])and consequently $\operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma)$ is surjective.

Clearly the composition

$$M \longrightarrow \operatorname{Div}_{T_N}(X_\Sigma) \longrightarrow \operatorname{Cl}(X_\Sigma)$$

is zero. Suppose $D \in \operatorname{Div}_{T_N}(X_\Sigma)$ maps to 0, i.e. $D = \operatorname{div}(f)$. Then, since the support of D is in $\bigcup_{\rho\in\Sigma(1)} D_\rho$, $\operatorname{div}(f) = 0$ on T_N and $f\colon T_N \cong \mathbb{C}[M] \to \mathbb{C}^*$ is a morphism by [Har77, 6, Prop. 6.3A, p. 132], so $f = c\chi^m$ with $c \in \mathbb{C}^*$, $m \in M$. It follows that $D = \operatorname{div}(c\chi^m) = \operatorname{div}(\chi^m)$.

To see that $m \to \operatorname{div}(\chi^m)$ is injective, suppose $\operatorname{div}(\chi^m) = 0$, i.e. $\langle m, u_\rho\rangle = 0$, $\forall f \in \Sigma(1)$, then $m = 0$ since u_ρ span $N_\mathbb{R}$ (this is where we use that X_Σ has no torus factors). □

Observation 2.4.5. $\mathrm{Cl}(X_\Sigma)$ *is a finitely generated Abelian group, i.e.*

$$\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}^l \times H \ \textit{for}\ H \cong \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{\alpha_r}}\ \textit{finite.}$$

Corollary 2.4.6. *In particular, for a normal toric variety X_Σ with no torus factors any divisor is linearly equivalent to a torus-invariant divisor, i.e. $D \in \mathrm{Div}(X_\Sigma)$, $[D] = [D_\rho]$ in $\mathrm{Cl}(X_\Sigma)$ for $\rho \in \Sigma(1)$.*

Example 2.4.7. Theorem 2.4.4 makes it extremely easy to compute classes of divisors in toric varieties. Consider for instance $X_\Sigma = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ with $k > 0$ as in Figure 1.2 (i). The map

$$A\colon M \cong \mathbb{Z}^2 \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \cong \mathbb{Z}^3, \qquad A = \begin{pmatrix} -1 & k \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its cokernel is one-dimensional since

$$\mathrm{Coker}(A) = \mathrm{Ker}(A^T) = \{(x, y, z) \in \mathbb{Z}^3 \ : \ -x + z = 0, kx + y = 0\} \cong \mathbb{Z},$$

so $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}$. Another way of seeing this is by linear equivalence of divisors. We take the standard basis $\{e_1, e_2\}$ for $M = \mathbb{Z}^2$ as in Figure 1.2 and compute:

$$\begin{aligned} 0 &\sim \mathrm{div}(\chi^{e_1}) = \langle e_1, u_1\rangle D_1 + \langle e_1, u_2\rangle D_2 + \langle e_1, u_3\rangle D_3 = D_1 - D_2, \\ 0 &\sim \mathrm{div}(\chi^{e_2}) = \langle e_2, u_1\rangle D_1 + \langle e_2, u_2\rangle D_2 + \langle e_2, u_3\rangle D_3 = kD_2 + D_3. \end{aligned}$$

So in $\mathrm{Cl}(X_\Sigma)$, $[D_1] = [D_2]$, $[D_3] = -k[D_2]$ and $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}\langle [D_1]\rangle \cong \mathbb{Z}$.

Theorem 2.4.8. *If the fan Σ of X_Σ has all its primitive generators lying on a hyperplane of $N_\mathbb{R}$, then X_Σ is Calabi-Yau.*

Proof. Let D_i be all the toric divisors with corresponding primitive generators u_i. If all the primitive generators lie on a hyperplane, then we can choose a basis for N such that the first coordinate of each of them is 1, i.e. $u_{in} = 1$. But then:

$$0 \sim \mathrm{div}(\chi^{e_1}) = \langle e_1, u_1\rangle D_1 + \cdots + \langle e_1, u_r\rangle D_r = \sum D_i.$$

Hence, since X_Σ is toric, [CLSar, Thm. 8.2.3] implies that

$$\mathcal{K} \cong \mathcal{O}_{X_\Sigma}(-\sum D_i) \cong \mathcal{O}_{X_\Sigma},$$

where $\mathcal{O}_X(D)$ is the sheaf of the Weil divisor D. Consequently the canonical divisor is trivial and X_Σ is Calabi-Yau. □

Chapter 3

Toric varieties as good quotients.

3.1 GIT preliminaries

One of the nicest features of toric varieties is that they can be seen as GIT quotients of a Zariski open subset of $\mathbb{C}^r$ modulo a group action. Explaining this construction is the main goal of this section. Geometric Invariant Theory (GIT) is a subject in its own right so we briefly recall what we understand as GIT quotients.

Let G be a **reductive group** (i.e. an affine algebraic group whose maximal connected solvable subgroup is a torus). Examples of these include finite groups, tori, semisimple groups or direct products of any of those.

Let $X = \mathrm{Spec}(R)$ be an affine variety and suppose that G **acts algebraically** on X, i.e. the map

$$\phi_g \colon X \longrightarrow X; \qquad x \longmapsto g \cdot x$$

is a morphism and moreover, since G is an algebraic group,

$$G \times X \longrightarrow X; \qquad (g, x) \longmapsto \phi_g(x)$$

is also a morphism.

We have an *induced action* of G on R induced by $\phi_g^* \colon R \longrightarrow R$:

$$g \cdot f = \phi_{g^{-1}}^*(f), \qquad (g \cdot f)(x) = f(g^{-1} \cdot x),\ \forall x \in X.$$

The **ring of invariants** is $R^G = \{f \in R : \ g \cdot f = f,\ \forall g \in G\}$ and the **set of G-orbits** $X/G = \{G \cdot x : \ \forall x \in X\}$.

The goal of GIT is to decide whether $X/G \cong \mathrm{Spec}(R^G)$, i.e. whether the set of orbits can be made into an affine variety. Unfortunately this is not true in general. In the case of reductive groups, the closest we can get to is [MFK94, Thm. 1.1]:

$$\{\text{closed } G\text{-orbits in } X\} \cong \mathrm{Spec}(R^G). \tag{3.1}$$

Let $\pi \colon X \longrightarrow Y$ be a surjective morphism where X and Y are abstract varieties. Let G act algebraically on X. If π is constant on G-orbits with $Y \cong \{\text{closed } G\text{-orbits in } X\}$, then we say that π is a **good categorical quotient**. There are further technical points that we have omitted (see [MFK94] for a detailed exposition) but this pseudo-definition serves our purposes. The usual notation for a good categorical quotient of X by G is

$$\pi \colon X \longrightarrow X/\!/G.$$

When G is reductive the quotient is always a good categorical one, and the projection $\pi\colon \operatorname{Spec}(R) \longrightarrow \operatorname{Spec}(R^G) \cong \operatorname{Spec}(R)//G$ as in (3.1) is just the case where X is affine.

In fact, it is enough to guarantee a good categorical quotient that isomorphism (3.1) holds for each open set in an affine open cover of Y (see [MFK94, Rmk. (5) p. 8]).

If all the G-orbits in X are closed then $X//G \cong X/G$ and we say that

$$\pi\colon X \longrightarrow X/G$$

is a **good geometric quotient**. Note that G being reductive does not guarantee that the quotient is a good geometric one.

3.2 Toric varieties as GIT quotients

As we will prove in this section normal toric varieties X_Σ can be expressed as good categorical quotients of the form:

$$X_\Sigma = \mathbb{C}^r \setminus \mathbf{Z}(\Sigma)//G, \qquad r = |\Sigma(1)|,$$

where G is reductive and $\mathbf{Z}(\Sigma)$ is a union of spaces $\mathbf{V}(x_{i_1}, \dots, x_{i_k})$.

First, given a generic toric variety we will define the group G, then $\mathbf{Z}(\Sigma)$ and see how $X_{\widetilde{\Sigma}} := \mathbb{C}^r \setminus \mathbf{Z}(\Sigma)$ is a normal toric variety for some fan $\widetilde{\Sigma}$. We will see how G is a reductive group that acts on $\widetilde{X_\Sigma}$. Finally we will prove that X_Σ is isomorphic to the set of closed orbits of $X_{\widetilde{\Sigma}}$.

Let $\widetilde{N} = \mathbb{Z}^{\Sigma(1)}$ and $\{e_\rho \,:\, \rho \in \Sigma(1)\}$ be the standard basis for $\widetilde{N}$. As usual, we assume that X_Σ has no torus factors (i.e. ρ spans $N_\mathbb{R}$ as a vector space). Hence, we can apply $\operatorname{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to the exact sequence in (2.5) to obtain

$$1 \to \operatorname{Hom}_\mathbb{Z}(\operatorname{Cl}(X_\Sigma), \mathbb{C}^*) \to \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \to \operatorname{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \to 1, \tag{3.2}$$

which is exact due to the fact that $\operatorname{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ is left exact. Right-exactness follows from the fact that $\mathbb{C}^*$ is a divisible group and therefore injective as a $\mathbb{Z}$-module. We use the convention $\operatorname{Hom}_\mathbb{Z}(0, \mathbb{C}^*) = 1$.

Define

$$G := \operatorname{Hom}_\mathbb{Z}(\operatorname{Cl}(X_\Sigma), \mathbb{C}^*) \leq \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\Sigma(1)}. \tag{3.3}$$

Hence, (3.2) becomes an exact sequence of affine algebraic groups:

$$1 \longrightarrow G \longrightarrow T_{\widetilde{N}} \longrightarrow T_N \longrightarrow 1 \tag{3.4}$$

where the second map sends $e_\rho \otimes t$ to $u_\rho \otimes t$ or in terms of (3.2) $f \in \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*)$ to $f \circ \operatorname{div} \in \operatorname{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$. This automatically implies the following lemma.

Lemma 3.2.1. *Given a basis $\{e_1, \dots, e_n\}$ for M, G can be expressed*

$$\begin{aligned} G &= \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \,:\, \prod_\rho t_\rho^{\langle m, u_\rho\rangle} = 1 \ \forall m \in M\} \\ &= \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \,:\, \prod_\rho t_\rho^{\langle e_i, u_\rho\rangle} = 1 \text{ for all } 1 \leq i \leq n\}. \end{aligned}$$

Lemma 3.2.2 ([CLSar, Lemma 5.11])**.** *G is the character group of* $\operatorname{Cl}(X_\Sigma)$.

Given $\Sigma \subseteq N$ we construct the fan $\widetilde{\Sigma} \subseteq \widetilde{N}_{\mathbb{R}}$ as

$$\widetilde{\Sigma} = \{\widetilde{\sigma} \ : \ \sigma \in \Sigma\}, \qquad \widetilde{\sigma} = \text{Cone}(e_\rho \ : \ \rho \in \sigma(1)) \subseteq \widetilde{N}_{\mathbb{R}} = \mathbb{R}^{\Sigma(1)}.$$

Furthermore, define the fan $\widetilde{\Sigma}_0$ consisting of the cone

$$\sigma_0 = \text{Cone}(e_\rho \ : \ \rho \in \Sigma(1))$$

and its faces. The fan $\widetilde{\Sigma}$ is a subfan of $\widetilde{\Sigma}_0$ containing some faces of σ_0.

Definition 3.2.3. The **exceptional set** of a fan $\Sigma \subseteq N$ is

$$\mathbf{Z}(\Sigma) = \bigcup_C \mathbf{V}(x_\rho \ : \ \rho \in C) \subseteq \mathbb{C}^{\Sigma(1)},$$

where $C \subseteq \Sigma(1)$ is all possible subsets of $\Sigma(1)$ which do not share a cone σ in Σ.

Proposition 3.2.4. *For a fan $\Sigma \subseteq N_{\mathbb{R}}$ and $\widetilde{\Sigma} \subseteq \widetilde{N}_{\mathbb{R}}$ as above*

(i) $X_{\widetilde{\Sigma}} = \mathbb{C}^{\Sigma(1)} \setminus \mathbf{Z}(\Sigma)$.

(ii) $e_\rho \mapsto u_\rho$ defines a map of lattices $\widetilde{\pi}\colon \widetilde{N} \to N$ compatible with $\widetilde{\Sigma}$ and Σ, inducing a toric morphism $X_{\widetilde{\Sigma}} \to X_\Sigma$ constant on G-orbits.

Proof. $\widetilde{\Sigma}_0$ is the fan of $\mathbb{C}^{\Sigma(1)}$ and the toric variety of $\widetilde{\Sigma}$ is, by the Orbit-Cone Correspondence (Theorem 2.3.5), given by $\mathbb{C}^{\Sigma(1)}$ without the orbits corresponding to cones in $\widetilde{\Sigma}_0 \setminus \widetilde{\Sigma}$, i.e. those which do not share a cone. The union of the deleted orbits is precisely the exceptional set $\mathbf{Z}(\Sigma)$. This proves (i).

The map of lattices is clearly compatible with $\widetilde{\Sigma}$ and Σ since $\widetilde{\sigma}$ gets mapped to σ. The correspondent map of tori is $\pi\colon T_{\widetilde{N}} \to T_N$, the same as in (3.4), so by the definition of G in (3.3) and the fact that π is a homomorphism, we have

$$\pi(g \cdot x) = \pi(g) \cdot \pi(x) = \pi(x) \quad \forall x \in X_{\widetilde{\Sigma}}, \ \forall g \in G \subseteq (\mathbb{C}^*)^{\Sigma(1)}.$$

□

Lemma 3.2.5. *For the group G defined from X_Σ as in* (3.3)*:*

(i) G is reductive.

(ii) G acts on $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{Z}(\Sigma)$.

Proof. By (3.3) and Observation 2.4.5 we have

$$G = \text{Hom}_{\mathbb{Z}}(\text{Cl}(X_\Sigma), \mathbb{C}^*) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^l \times H, \mathbb{C}^*) \cong (\mathbb{C}^*)^l \times \text{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*),$$

which is the product of a torus and a finite Abelian group and therefore reductive, proving (i).

The torus $(\mathbb{C}^*)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ by coordinatewise multiplication[1] and since $\mathbf{Z}(\Sigma)$ is a union of coordinate spaces, the torus leaves $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{Z}(\Sigma)$ invariant. Since $G \leq (\mathbb{C}^*)^{\Sigma(1)}$ the same is true for G. □

[1]Often called diagonal matrix multiplication.

The given method to obtain this quotient is sometimes referred to as Cox's construction after David Cox who originally published the following two theorems in [Cox95]. However, as Cox himself mentions in his article, versions of these theorems (for instance for symplectic manifolds) were proved around the same time in [Aud91], [Mus94].

In the case of simplicial fans, the quotient becomes geometrical. The original proof of theorems 3.2.6 and 3.2.7 can be found in [Cox95, Theorem 2.1]. A clearer version using more modern notation can be found in [CLSar, Theorem 5.1.10].

Theorem 3.2.6. *Let X_Σ be a toric variety without torus factors and G as in* (3.3)*. Then the map $\pi\colon X_{\widetilde{\Sigma}} \longrightarrow X_\Sigma$ in Proposition 3.2.4 is a good categorical quotient for the action of G on $X_{\widetilde{\Sigma}}$, i.e.*

$$X_\Sigma \cong X_{\widetilde{\Sigma}}//G = (\mathbb{C}^{\Sigma(1)} \setminus \mathbf{Z}(\Sigma))//G.$$

Proof. First, note that we can reduce to the case of a single cone, since it is enough to find a GIT quotient for each open subset of an affine cover, as long as G acts on all of them. A natural choice of affine cover will be U_σ where $\sigma \in \Sigma$.

Therefore we need to show that

$$\pi_\sigma := \pi|_{\pi^{-1}(U_\sigma)}\colon \pi^{-1}(U_\sigma) \longrightarrow U_\sigma$$

is a good categorical quotient for the group G. Note that $\pi^{-1}(U_\sigma) = U_{\widetilde{\sigma}}$ for $\widetilde{\sigma} \subset \mathbb{Z}^{\Sigma(1)}$. We will work with coordinate rings instead of with Zariski open sets. The semigroup of $\widetilde{\sigma}$ is:

$$S_{\widetilde{\sigma}} := \widetilde{\sigma}^\vee \cap \mathbb{Z}^{\Sigma(1)} = \{(a_\rho) \in \mathbb{Z}^{\Sigma(1)} \;:\; a_\rho \geq 0 \text{ for } \rho \in \sigma(1)\}.$$

The coordinate ring

$$R := \mathbb{C}[S_{\widetilde{\sigma}}] = \mathbb{C}[\prod_\rho x_\rho^{a_\rho} \;:\; a_\rho \geq 0 \;\forall \rho \in \sigma(1)] = \mathbb{C}[x_\rho \;:\; \rho \in \Sigma(1)]_{x^{\widetilde{\sigma}}},$$

is the localisation of the affine coordinate ring of $\mathbb{C}^r$, $r = |\Sigma(1)|$ at $x^{\widetilde{\sigma}} = \prod_{\rho \notin \sigma(1)} x^\rho$. Hence, the map $\pi^*\colon \mathbb{C}[U_{\widetilde{\sigma}}] \to \mathbb{C}[U_\sigma]$ is induced by

$$M \to \mathbb{Z}^{\Sigma(1)} \quad m \mapsto (|m, u_\rho|) \tag{3.5}$$

by Theorem 1.4.5, and $|m, u_\rho| \geq 0$, for all $\rho \in \sigma(1)$ and all $m \in \widetilde{\sigma} \cap \mathbb{Z}^{\Sigma(1)}$.

By Proposition 3.2.4 (ii), G acts on $X_{\widetilde{\sigma}}$ (and therefore on R) and it is constant in G-orbits. Together with the fact that G is reductive (Lemma 3.2.5) we obtain that

$$\Pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \longrightarrow R^G \subseteq R \tag{3.6}$$

induces a good categorical quotient:

$$U_{\widetilde{\sigma}} \longrightarrow \operatorname{Spec}(R^G) = U_{\widetilde{\sigma}}//G.$$

Finally, we need to prove $\mathbb{C}[\sigma^\vee \cap M] \cong R^G$ via Π_σ^* as in (3.5) and (3.6). This implies $U_\sigma \cong U_{\widetilde{\sigma}}//G$.

For injectivity take $\chi^m, \chi^{m'} \in \mathbb{C}[\sigma^\vee \cap M]$ and suppose that they map to the same element via Π_σ^*:

$$\begin{aligned} & \Pi^*(\chi^m) = \Pi^*(\chi^{m'}) \\ \Leftrightarrow \quad & \prod_{\rho\in\Sigma(1)} t_\rho^{\langle m,u_\rho\rangle} = \prod_{\rho\in\Sigma(1)} t_\rho^{\langle m',u_\rho\rangle} \\ \Leftrightarrow \quad & \langle m,u_\rho\rangle = \langle m',u_\rho\rangle \quad \forall\rho\in\Sigma(1) \\ \Leftrightarrow \quad & m = m'. \end{aligned}$$

For surjectivity let $f = \Sigma_\alpha c_\alpha x^\alpha \in R^G$. Since $R^G \subseteq R$, $x^\alpha = \prod_\rho x_\rho^{a_\rho}$ where $a_\rho \geq 0$ for all $\rho \in \sigma(1)$. Moreover, since f is G-invariant, for any $(t_\rho^\alpha) \in G$ we get

$$\sum_\alpha c_\alpha x^\alpha = \sum_\alpha c_\alpha t^\alpha x^\alpha$$

where t^α is the character of $T = (t_\rho) \in G$ and $t^\alpha = 1$ whenever $c_\alpha \neq 0$. By Lemma 3.2.2, those t_α are the characters of $\mathrm{Cl}(X_\Sigma)$ and therefore since they are trivial as characters, by (2.5), $\alpha = (\alpha_\rho)$ come from elements $m \in M$, i.e. $\alpha = \langle m, u_\rho\rangle$, for all $\rho \in \Sigma(1)$. Therefore, since $x^\alpha \in R$ we get that

$$\langle m, u_\rho\rangle = a_\rho \geq 0 \quad \forall\rho\in\sigma(1),$$

so Π_σ^* is surjective. □

Theorem 3.2.7. *If X_Σ is a toric variety with no torus factors and Σ is simplicial then*

$$X_\Sigma \cong (\mathbb{C}^{\Sigma(1)} \setminus \mathbf{Z}(\Sigma))/G,$$

i.e. $X_{\widetilde{\Sigma}} \longrightarrow X_\Sigma$ is a good geometric quotient for the action of G on $X_{\widetilde{\Sigma}}$.

Example 3.2.8. We now show how the fan in Figure 1.2 b) of Example 1.3.1 gives the variety $\mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(m))$ for $m > 0$ using the GIT construction. The case $m < 0$ is analogous. The subsets of rays not generating a cone are $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$, but since one is contained in another we just need to consider the *smallest* (the order of inclusion is reversed when we pass to the subvarieties) i.e.:

$$\mathbf{Z}(\Sigma) = \mathbf{V}(z_0, z_1) \cup \mathbf{V}(z_0, z_1, \theta) = \mathbf{V}(z_0, z_1)$$

(We assign the coordinate θ to v_3 to agree with the notation in example 1.3.1). The second map in the exact sequence (3.4) is given by:

$$\phi\colon (\mathbb{C}^*)^3 \longrightarrow (\mathbb{C}^*)^2 \qquad \phi(t_1,t_2,t_3) = (t_1 t_2^{-1}, t_2^{-m} t_3)$$

and therefore $\phi(t_1, t_2, t_3) = (1, 1) \Leftrightarrow t_1 = t_2,\ t_2^m = t_3$. This can also be seen using lemma 3.2.1. Hence, $G = \mathrm{Ker}(\phi) = \langle(t, t, t^m)\rangle$, and the variety is:

$$X_\Sigma = \frac{\mathbb{C}^3 \setminus \mathbf{V}(z_0, z_1)}{\langle(t,t,t^m)\rangle}.$$

Note that the quotient is geometric since the fan is simplicial.The group action identifies the points $(z_0, z_1, \theta) \sim (z_0 t, z_1 t, \theta t^m)$. Since $z_0 z_1 \neq 0$, we can see the points as $(1, \frac{z_1}{z_0}, \frac{\theta}{z_0^m})$ when $z_0 \neq 0$ and in a similar way when $z_1 \neq 0$. Giving values as in equation (1.4) of Example 1.3.1 we get the transition functions in (1.5).

Note we took an arbitrary ordering for the rays. A different ordering would give a permutation of the coordinates.

3.3 T-invariant subvarieties

We prove a version of the Orbit-Cone correspondence as well as the dimension statement in Theorem 2.3.5 (i) using the GIT quotient. Let $\sigma \in \Sigma$ be a simplicial cone generated from the semigroup of rays $\rho_1, \ldots, \rho_k$. Note that

$$Z_\sigma = \{x \in X_\Sigma | x_{\rho_1} = \ldots = x_{\rho_k} = 0\}/G = \widetilde{Z_\sigma}/G \tag{3.7}$$

has no points in $\mathbf{Z}(\Sigma)$ since the rays generate a cone and therefore it is a codimension k-subvariety, T-invariant by construction.

Theorem 3.3.1. *Let Σ be a simplicial fan. Then, the assignment $\sigma \longrightarrow Z_\sigma$ gives an order reversing correspondence:*

$$\{\textit{cones in the fan}\} \overset{1:1}{\longleftrightarrow} \{\textit{non empty } T\textit{-invariant subvarieties}\}.$$

Proof. The map from cones to varieties is given by (3.7) and the order reversing is clear (adding one ray increases the codimension by 1).

Given a T-invariant subvariety $Z \subseteq X_\Sigma$, it arises from $\widetilde{Z} \subseteq X_{\widetilde{\Sigma}}$ quotient out by G. Since Σ is simplicial, by Theorem 3.2.7 the quotient is geometric. For $\widetilde{Z}$ to be invariant by G and $T_{\widetilde{N}}$, it has to be a space $\widetilde{Z} = \mathbf{V}(x_{i_1}, \ldots, x_{i_r}) \subseteq \mathbb{C}^r$ where $r = |\Sigma(1)|$ by the coordinate expression of G in Lemma 3.2.1. Thus $Z = \widetilde{Z}/G$ has the form in (3.7). □

Chapter 4

Resolution of singularities

Definition 4.0.2. Given $\Sigma, \Sigma' \subseteq N_{\mathbb{R}}$, we say that $\Sigma' \subseteq N_{\mathbb{R}}$ **refines** Σ if $\mathrm{Id}\colon (N_{\mathbb{R}}, \Sigma') \to (N_{\mathbb{R}}, \Sigma)$ is compatible with Σ', Σ and their supports are the same. By Theorem 1.4.6 Id induces a toric map $X_{\Sigma'} \to X_{\Sigma}$.

Example 4.0.3. Figure 4.1 shows on the right side a simplicial, non-smooth fan $\Sigma = \langle \mathrm{Cone}(e_1, 2e_2 + e_1) \rangle$ and its refinement $\Sigma' = \langle \mathrm{Cone}(e_1, e_1 + e_2), \mathrm{Cone}(e_1 + e_2, 2e_2 + e_1) \rangle$

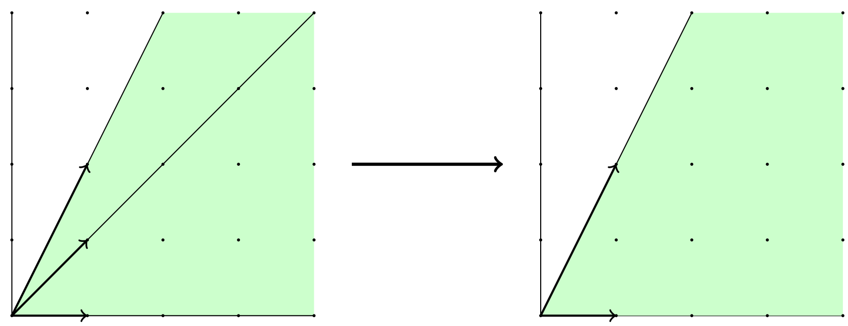


Figure 4.1: Fans of $\mathbb{C}^2/\mathbb{Z}_2$ after and before blowing up at 0.

In Proposition 2.2.1 we saw that for a normal toric variety X_{Σ} to be smooth, every cone σ, the intersection $\sigma \cap N$ must be generated by a $\mathbb{Z}$ basis. Otherwise extra vectors are required to generate its affine patch. The way singularities arise in this situation was illustrated by example 2.2.2. An obvious way of smoothing out σ is to refine it by adding rays ρ_{v_i} generated by these extra vectors. The resulting fan Σ' will be smooth and we have a map $X_{\Sigma'} \to X_{\Sigma}$. This is the general philosophy underlined in what follows. We will refine the fan by turning the non-simplicial cones into simplicial and then the non-smooth cones in the fan into smooth. The advantage of this approach is that it is purely algorithmical and combinatorial, *constructing* the resolution.

Definition 4.0.4. Given a fan $\Sigma \subseteq N_{\mathbb{R}}$ and a primitive element $v \in |\Sigma| \cap N \setminus \{0\}$, the **star subdivision** of Σ at v, $\Sigma^*(v)$ is the following set of cones:

(i) σ where $v \notin \sigma \in \Sigma$.

(ii) $\mathrm{Cone}(\tau, v)$ where $v \notin \tau \in \Sigma$ and $\tau \cup \{v\} \subseteq \sigma \in \Sigma$.

The following lemma is intuitive, but its proof (see [CLSar, Lem. 11.1.3]) is somewhat technical and lengthy:

Lemma 4.0.5. *$\Sigma^*(v)$ is a fan, and moreover, it is a refinement of Σ. The 1-dimensional cones of $\Sigma^*(v)$ are the 1-dimensional cones of Σ plus the ray $\rho_v = \text{Cone}(v)$ generated by v, i.e.*

$$\Sigma^*(v)(1) = \Sigma(1) \cup \{\rho_v\}.$$

Example 4.0.6. The fan Σ' in example 4.0.3 is the star subdivison of Σ at $v = (1,1)$, i.e $\Sigma' = \Sigma^*(v)$.

The following lemma requires a lengthy analysis on $\mathbb{Q}$-Cartier divisors, proper and projective morphisms on toric varieties, so we refer the reader to the reference:

Lemma 4.0.7 ([CLSar, Prop. 11.1.6])**.** *The star subdivision $\Sigma^*(v)$ of an arbitrary fan Σ has the following properties*

(i) The prime divisor D_{ρ_v} is $\mathbb{Q}$-Cartier.

(ii) The induced toric morphism $X_{\Sigma^*(v)} \to X_\Sigma$ is projective.

The first step towards the desingularisation is to make the fan simplicial:

Proposition 4.0.8. *Every fan Σ has a refinement Σ' with the following properties*

(i) Σ' is obtained from Σ by a sequence of star subdivisions.

(ii) Σ' is simplicial.

(iii) Σ' contains every simplicial cone of Σ.

(iv) The induced toric morphism $X'_\Sigma \to X_\Sigma$ is projective.

Proof. We will obtain Σ' from Σ via star subdivisions preserving the edges. The other parts will follow. Let

$$A = \{\rho \in \Sigma(1) \ : \ D_\rho \text{ is not } \mathbb{Q}\text{-Cartier}\}.$$

If $|A| = 0$, then all divisors are $\mathbb{Q}$-Cartier and [CLSar, Prop. 4.27] implies that Σ is simplicial. If $|A| > 0$, pick ρ such that D_ρ is not $\mathbb{Q}$-Cartier and let u_ρ be its generator. Lemma 4.0.5 implies $\Sigma^*(u_\rho)(1) = \Sigma(1)$ since $\rho \in \Sigma(1)$ and Lemma 4.0.7 implies that D_{ρ_v} is $\mathbb{Q}$-Cartier and $X_{\Sigma^*(u_\rho)} \to X_\Sigma$ is projective. By finite induction we repeat this procedure until we obtain the required Σ' satisfying parts (i) and (ii). (iv) follows since the composition of projective morphisms is projective. For (iii), let $\sigma' \subset \Sigma'$ be mapped into σ simplicial. Since $\Sigma(1) = \Sigma'(1)$, $\sigma'(1) \subseteq \sigma(1)$ and σ' is a face of σ, since σ is simplicial. But

$$\sigma = \bigcup_{\substack{\sigma' \in \Sigma' \\ \sigma' \subseteq \sigma}} \sigma',$$

so $\sigma = \sigma'$ for some $\sigma' \in \Sigma'$. □

Before subdividing simplicial fans into smooth ones, we need a tool to measure how far a simplicial fan is from being smooth.

Definition 4.0.9. Let $\sigma = \text{Cone}(u_1, \ldots, u_d) \subset N_\mathbb{R}$ be a simplicial cone and $N_\sigma = \text{Span}(\sigma) \cap N$, then the **multiplicity** of σ is

$$\text{mult}(\sigma) = [N_\sigma : \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_d],$$

and the multiplicity of a fan Σ is

$$\text{mult}(\Sigma) = \max_{\sigma \in \Sigma} \text{mult}(\sigma).$$

Lemma 4.0.10 ([CLSar, Prop. 11.1.8])**.** *Let σ be a simplicial cone. Then, for u_i and N_σ as in definition 4.0.9, we have:*

(i) $\text{mult}(\sigma)$ *is the number of points in $P_\sigma \cap N$ where*

$$P_\sigma = \{\sum_{i=1}^{d} \lambda_i u_i \ : \ 0 \leq \lambda_i < 1\}$$

(ii) Let $e_1 \dots, e_d$ be a basis of N_σ and write $u_i = \sum a_{ij} e_j$. Then $\text{mult}(\sigma) = |\det(a_{ij})|$.

(iii) $\text{mult}(\tau) \leq \text{mult}(\sigma)$ *whenever τ is a face of σ.*

Example 4.0.11. The multiplicity of $\sigma = \text{Cone}(e_1, e_1 + 2e_2)$ as in the first cone in figure 4.1 is 2. Note a cone has a multiplicity of one if and only if it is smooth.

Theorem 4.0.12. *Every fan Σ has a refinement Σ' with the following properties:*

(i) Σ' is smooth.

(ii) Σ' contains every smooth cone of Σ.

(iii) Σ' is obtained from Σ by a sequence of star subdivisions.

(iv) The toric morphism $\phi \colon X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities.

Proof. Since all smooth cones are simplicial, we can apply Proposition 4.0.8 and assume Σ is simplicial. We will build Σ' by a sequence of star subdivisions of non-smooth cones until we make them smooth. Once we have done this, lemma 4.0.7 (ii), Proposition 4.0.8 (iv) and the fact that the composition of projective morphisms is projective will give us a projective resolution $\phi : X'_\Sigma \to X_\Sigma$. The fact that ϕ is an isomorphism on the smooth locus of X_Σ, as in definiton 0.0.4 is a consequence of part (ii) of this theorem and remark 1.1.10 (ii).

As in the proof of Proposition 4.0.8, we will do a finite induction, this time on $\text{mult}(\Sigma)$. The inductive step will be to find a subdivision $\Sigma^*(v)$ which leaves all smooth cones unaltered and satisfies either:

$$\begin{aligned} &\text{mult}\left(\Sigma^*(v)\right) \leq \text{mult}(\Sigma) \\ &\text{mult}\left(\Sigma^*(v)\right) \leq \text{mult}(\Sigma) \text{ but } \Sigma^*(v) \text{ has fewer cones with this multiplicity} \end{aligned}$$

Pick $\widetilde{\sigma} \in \Sigma$ with maximal multiplicity and pick $v \in P_{\widetilde{\sigma}} \cap N \setminus \{0\}$. According to the definition of star subdivision we replace all σ such that $v \in \sigma$ with $\text{Cone}(\tau, v)$ for τ a face of σ, $v \notin \tau$. Let $\widetilde{\tau}$ be such a face of $\widetilde{\sigma}$.

Claim. $\text{mult}(\text{Cone}(v, \widetilde{\tau})) < \text{mult}(\widetilde{\sigma})$.

Proof of the claim. By Lemma 4.0.10, $v = \sum_{i=1}^{d} \lambda_i u_i$, $0 < \lambda_i < 1$ for $\{u_i\}_{i=1}^{d}$, the ray generators of the minimal face $\widetilde{\tau}$ of $\widetilde{\sigma}$ containing v. Since $\widetilde{\tau}$ is a face of $\widetilde{\sigma}$ we can complete that base to $\{u_i\}_{i=1}^s$, generators for $\widetilde{\sigma}$, with $s > d$. But $v \notin \widetilde{\tau}$, so we have i such that $u_i \notin \tau$. Hence, by parts (iii) and (ii) in Lemma 4.0.10:

$$\begin{aligned} \text{mult}\,(\text{Cone}(\widetilde{\tau}, v)) &\leq \text{mult}\,(\text{Cone}(u_1, \ldots, \hat{u}_i, \ldots, u_s, v)) \\ &= |\det(u_1, \ldots, \hat{u}_i, \ldots u_s, \lambda_i u_i)| \\ &= \lambda_i |\det(u_1, \ldots, u_s)| = \lambda_i \text{mult}(\widetilde{\sigma}) < \text{mult}(\widetilde{\sigma}). \end{aligned}$$ □

We can finish the proof now. Since $\widetilde{\sigma}$ is not smooth, by remark 1.1.10 (ii) $\widetilde{\sigma}$ lies in no smooth cone of Σ, so all smooth cones of Σ are in $\Sigma^*(v)$ and we have lost at least one cone of maximal multiplicity, so this finishes the inductive step. □

The power of Theorem 4.0.12 is shown in the next chapter, when we resolve the conifold.

Chapter 5

Applications to String Theory

This chapter has two parts. First, we introduce the GLSM motivating why physicists care about toric varieties. Then we analyse a particular class of singular toric variety, the generalised conifold, and show how to resolve it by using the toric resolutions introduced in chapter 4. However, the generalised conifold happens to be Calabi-Yau and we will resolve it in such a way that the smooth space is also Calabi-Yau (this is called a crepant resolution).

The generalised conifold appears very frequently in the most recent literature in Theoretical Physics and it is one of the examples that mathematical physicists use to tackle Donaldson-Thomas type invariants, usually arising from a quiver algebra. In [Sze08] it is used to calculate non-commutative Donaldson-Thomas Invariants. In [Nag10], Nagao uses it to generalise to higher dimensional Donaldson-Thomas invariants and [NY09] uses it to work with the Topological Vertex. To compute these invariants it is usually necessary to understand what the crepant resolutions of the generalised conifold are made of. It turns out that they can be seen as multiple copies of the total space of bundles $\mathcal{O}(-1,-1)$ and $\mathcal{O}(-2,-0)$. In particular, the topological vertex, as defined in [AKMV05], is computed for the generalised conifold [IKP06] as:

$$Z_{\text{top}}(q,Q) = \left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}\right)^{\chi/2} \prod_{1\leq i<j\leq \chi-1} \prod_{k=1}^{\infty} (1 - Q_i \cdots Q_j q^k)^{s_i \cdots s_j k}, \tag{5.1}$$

for χ, the Euler characteristics of the generalised conifold, $Q_1, ..., Q_{\chi-1}$ are the Kähler moduli that measure the sizes of the embedded $\mathbb{CP}^1$s and the $s_i = -1$ or $+1$ for $\mathcal{O}(-1,-1)$ or $\mathcal{O}(-2,0)$, respectively. This is also used in [OSY10]. In section 5.2 we show that these are all the embeddings and how to count them.

5.1 GLSM and toric varieties

The main goal of this section is explaining from a mathematical point of view what the physics motivation for the study of toric varieties is. For a detailed treatement we refer the reader to [CK99, Appendix B].

The Gauge Linear Sigma Model (GLSM) is a two-dimensional gauge theory. This means that we have a 2-dimensional manifold (our space-time) with a graded principal $U(1)$-bundle over it. We will work in the absence of a superpotential. We will have s graded connections $V_1, \ldots, V_s$ called gauge superfields and we will operate with n chiral superfields $\Phi_1, \ldots, \Phi_n$ representing n particles with scalar values $\phi_1, \ldots, \phi_n$ (we require $n \geq s$. These particles will have different charges depending on which gauge superfield is acting on them. We will write $Q_{i,a}$ for the charge of Ψ_i with respect to V_a.

The equations of movement of these fields are determined by a Lagrangian that we will omit. Using it the potential energy can be written as:

$$U(\phi_i) = \Sigma_{a=1}^{s} \frac{e_a^2}{2}\left(\Sigma_{i=1}^{n} Q_{i,a}(\phi_i)^2 - r_a\right)^2, \qquad a = 1, \ldots, s \tag{5.2}$$

where e_a are the gauge coupling constants of each V_a and r_a, also known as the Fayet-Iliopoulos term (or FI parameter). The FI parameter is an integral of a function depending on V_a over the space-time. Suppose we are interested in the space of all possible supersymmetric ground states of this theory. We need to find the zeroes of the potential energy (5.2), i.e. solutions to:

$$\Sigma_{i=1}^{n} Q_{i,a}(\phi_i)^2 = r_a \qquad a = 1, \ldots, s. \tag{5.3}$$

For a generic choice of charges $Q_{i,a}$ and FI parameters (i.e. for s linearly independent polynomials as in (5.3)) , the space formed by all the possible ϕ_i is an $(n-s)$-dimensional affine variety.

Theorem 5.1.1 ([HKK$^+$03][Section 7.3])**.** *In the absence of a superpotential, and for a generic choice of charges $Q_{i,a}$ and an appropriate choice of FI parameters, the space of energy ground states of a 2-dimensional GLSM is an $(n-s)$-dimensional toric variety X_Σ whose fan Σ has n rays.*

Remark. A discussion on this result can be found in the reference. However, we will just state where the fan of 5.3 comes from. Since the choice of charges is generic, we can find principal vectors $\{v_1, \ldots, v_n\} \subseteq \mathbb{Z}^{n-s}$ satisfying

$$\sum_{i=1}^{n} Q_{i,a} v_i = 0 \qquad \forall a = 1, \ldots, s. \tag{5.4}$$

This vectors therefore generate $\Sigma(1)$. A construction similar to the one in chapter 3 is used and an exact sequence like 3.4 is constructed, where $\widetilde{N} = \mathbb{Z}^n$ with basis $\{e_1, \ldots, e_n\}$:

$$1 \longrightarrow G \longrightarrow T_{\widetilde{N}} \longrightarrow T_N \longrightarrow 1.$$

The map $G \to T_{\widetilde{N}}$ is defined as

$$(t_1, \ldots, t_s) \longmapsto (\prod_{a=1}^{s} t_a^{Q_{1,a}}, \ldots, \prod_{a=1}^{s} t_n^{Q_{n,a}}) \tag{5.5}$$

and $T_{\widetilde{N}} \to T_N$ sends $e_i \otimes z \to v_i \otimes z$. As in chapter 3, this only defines what the rays of Σ are. Whether they generate a cone or not depends on the choice of the r_a. Note, for instance, that for a GLSM with one single vector of charges $(1,1)$, the variety $|\phi_1|^2 + |\phi_2|^2 = r$ has no solutions for $r < 0$.

The charges provide certain intersection numbers of T-invariant subvarieties of X_Σ [HKK$^+$03, Chapter 7.4]. Also, the charges and the relation among them are the entries of a matrix which lets us work with Landau-Ginzburg models.

5.2 Application: Resolution of an affine toric conifold

Assume we work on the standard lattice $N = \mathbb{Z}^3$ with the standard inner product. Let $\Sigma \subseteq \mathbb{R}^3$ be the fan with a single cone σ generated by vectors given by the following matrix:

$$\begin{bmatrix} v_0 & v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & N_1 & N_0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where $N_0 \geq N_1$, $N_1 \geq 0$, $N_0 > 0$. See Figure 5.1 for the case $N_1 = 2$, $N_0 = 4$.

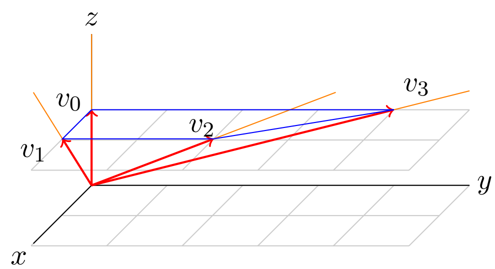


Figure 5.1: Fan of the conifold $\mathbf{V}(xy - z^{N_0} w^{N_1})$.

Remark. By Theorem 2.4.8 we already know that X_Σ is Calabi-Yau and by Theorem 5.1.1 we can see it as a Gauge Linear Sigma Model.

We want to find X_Σ as the spectrum of the algebra of the semigroup S_σ. To do so, we find the inner pointing vectors perpendicular to each facet, and choose the minimal ones within the lattice, for instance for the facet generated by $\{v_0,\ v_1\}$:

$$w_{0,1} \perp \langle v_0, v_1 \rangle \qquad w_{01} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Note that this method only works for dimension 3. For other dimensions we would use the Gram-Schmidt method.

Similarly, we find all the other inner pointing vectors w_{ij} perpendicular to facets generated by $\{v_i,\ v_j\}$, obtaining the matrix:

$$\begin{bmatrix} w_{01} & w_{12} & w_{23} & w_{30} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & N_1 - N_0 & 1 \\ 0 & 1 & N_0 & 0 \end{bmatrix},$$

which gives rise to the dual cone in Figure 5.2.

The only relation on the dual cone is:

$$w_{01} - N_0 w_{12} + w_{23} - N_1 w_{30} = 0,$$

so the resulting coordinate ring and variety are:

$$\mathbb{C}[S_\sigma] \cong \frac{\mathbb{C}[x, y, z, w]}{\langle xy - z^{N_0} w^{N_1} \rangle}, \qquad X_\Sigma = X_\sigma = \mathbf{V}(xy - z^{N_0} w^{N_1}).$$

This variety has a singularity at $(0, 0, 0, 0)$, known in the physics literature as a conifold

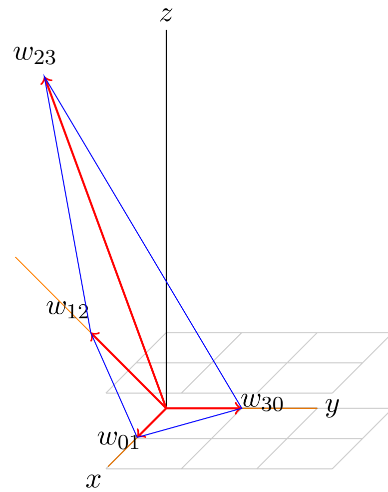


Figure 5.2: Dual cone of the conifold $\mathbf{V}(xy - z^{N_0}w^{N_1})$.

singularity. For $N_0 > 1$ it is also singular at $(0, 0, 0, w)$, $w \in \mathbb{C}$ and for $N_1 > 1$ at $(0, 0, z, 0)$, $z \in \mathbb{C}$. The variety itself is usually called conifold and many times the values N_0, N_1 are taken to be 1.

We would like to resolve the singularity so that the resolution is both smooth, Calabi-Yau and toric. Since there is no other point in the interior of $\{z = 1\} \cap \sigma \cap N$ the only way we can resolve the singularity preserving the Calabi-Yau condition is by crepant resolutions. This means dividing the fan Σ into a new fan $\widetilde{\Sigma}$ in which all the cones are generated by a basis for $\mathbb{R}^3$ of elements in $\sigma \cap N$. To remove the singularity completely and preserve the triviality of the canonical bundle, we need to add vectors to all the points in $\sigma \cap \{z = 1\} \cap N$ and facets between some of them so that all the cones in $\widetilde{\Sigma}$ are generated by 3 vectors. This procedure follows from the algorithm described in Chapter 4, where we repeatedly take $\sigma^*(v)$, the star subdivision of σ for v a minimal integral generator in a facet of σ, forcing that $v \in \{z = 1\}$. First we turn the cone into a simplicial cone, by joining two of its rays, and then we subdivide each of the cones until they are smooth. Figure 5.3 shows two possible *triangulations*. The first resolution which turns the fan into a simplicial one appears thicker. They are 2-dimensional images since they correspond to the intersection with the plane $\{z = 1\}$. Note that the resolution is crepant.

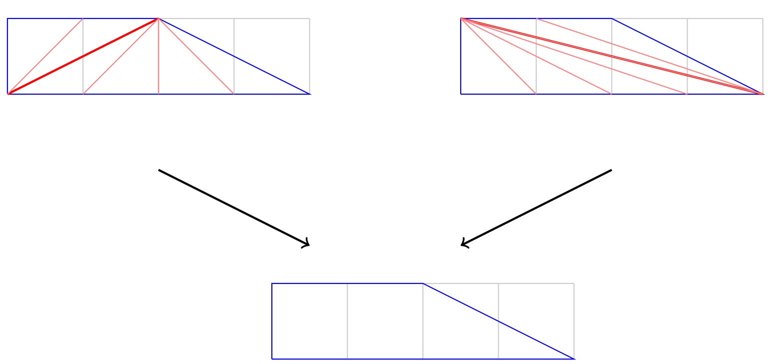


Figure 5.3: Examples of triangulations for the fan of the conifold $\mathbf{V}(xy - z^{N_0}w^{N_1})$.

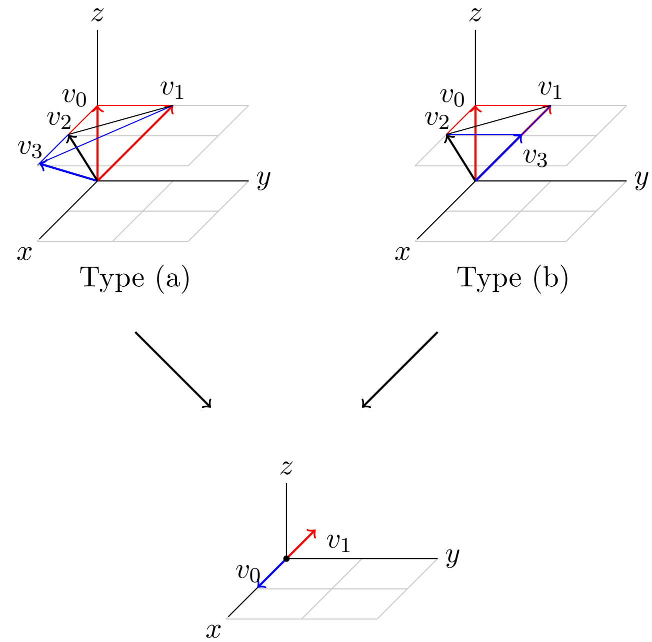


Figure 5.4: Basic small resolutions for the conifold.

Suppose we have subdivided Σ with a triangulation such that the resulting fan $\widetilde{\Sigma}$ corresponds to a smooth variety, i.e. for each cone σ, $\sigma \cap N$ must be generated by the primitive generators of 3 rays. Now consider the cones in Figure 5.4, and their projection over the cones of $\mathbb{CP}^1$.

Since the preimages of cones in the fans downstairs in Figure 5.4 under the projections are cones (note the colours), this map is toric, and since they are trivial over open sets, both triangulations must be vector bundles over $\mathbb{CP}^1$ (they are smooth since their cones are simplicial).

In fact all the possible triangulations of Σ can be modelled as a union of the two types (a) and (b) in Figure 5.4. To see this, note that all the tetrahedra which determine cones in $\widetilde{\Sigma}$ have volume

$$V = \frac{1}{3} A_0 h = \frac{1}{6}.$$

where $A_0 = \frac{1}{2}$ is the base at $\{z = 1\}$ and $h = 1$ is the height. This is clearly the case for all tetrahedra in $\widetilde{\Sigma}$.

Now given two cones σ_1, σ_2 sharing a facet, their tetrahedra will have three vertices in common: the origin and, say, u and v. The other two vertices, c_1 and c_2, can either have the same coordinate y (0 or 1), or a different one. Now we can apply to both cones some map ϕ in $SL(3, \mathbb{Z})$ (an isotropy) to go to a situation of type (a) or (b) as in Figure 5.4. This induces a morphism on the corresponding toric varieties. Since ϕ is invertible, the map in the varieties is an isomorphism. Note that the types (a) and (b) are not equivalent by $SL(3, \mathbb{Z})$ since one of the tetrahedra should be fixed but not the other one.

Proposition 5.2.1. *For the fans in Figure 5.4*

(i) The variety for the fan of type (a) is $\mathcal{O}_{\mathbb{CP}^1}(-2, 0) := \mathcal{O}_{\mathbb{CP}^1}(-2) \oplus \mathcal{O}_{\mathbb{CP}^1}$.

(ii) The variety for the fan of type (b) is $\mathcal{O}_{\mathbb{CP}^1}(-1, -1) := \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$.

Proof. For part (i), note that the fan is generated by the vectors

$$\begin{bmatrix} v_0 & v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

generating the cones $\langle v_0, v_1, v_2 \rangle$ and $\langle v_1, v_2, v_3 \rangle$. The dual cones are generated by the vectors

$$\begin{bmatrix} w_{01} & w_{02} & w_{12} & w_{13} & w_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

Note the relations among these vectors:

$$w_{12} + w_{01} = w_{13}, \qquad w_{12} + w_{02} = w_{32}, \qquad -w_{12} = (-1)w_{12}.$$

We therefore have two affine patches isomorphic to $\mathbb{C}^3$ with coordinates U, V, X corresponding to w_{01}, w_{02}, w_{12} and $\widetilde{U}, \widetilde{V}, \widetilde{W}$ corresponding to $w_{13}, w_{32}, -w_{12}$. The transition functions between the two patches are determined by:

$$\widetilde{X} = X^{-1}, \qquad UX = \widetilde{U}, \qquad VX = \widetilde{V},$$

in the intersection of both (i.e, at $X, \widetilde{X} \neq 0$)). This is the local description of the desired bundle.

For part (ii), note that the fan is generated by the vectors

$$\begin{bmatrix} v_0 & v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

grouped in two cones as before. The dual cones are generated by the same vectors as before but with different values:

$$\begin{bmatrix} w_{01} & w_{02} & w_{12} & w_{13} & w_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}.$$

The relations are now:

$$2w_{12} + w_{01} = w_{13}, \qquad w_{02} = w_{32}, \qquad -w_{12} = (-1)w_{12}.$$

And the transition functions are those of $\mathcal{O}_{\mathbb{CP}^1}(-2, 0)$:

$$\widetilde{X} = X^{-1}, \qquad \widetilde{V} = V, \qquad \widetilde{U} = X^2 U. \qquad \square$$

As we mentioned at the beginning, for the formula of the topological vertex (5.1), it is important to be able to determine which is each of the resolutions:

Algorithm 5.2.2. *Consider a toric projective resolution of singularities which is crepant for the generalised conifold, as above. Let* `l[i]` *be the lines of the resolution, with origin* `l[i].o` *and end* `l[i].e` *where we take the criteria that the origin has component* $y = 0$ *and the end* $y = 1$ *and* `i` *ranking between* 0 *and* n. *Then, for* `i` *taking values from* 1 *to* $n - 1$ *the values of* s_i *in* (5.1) *are determined as follows:*

```
If ((l[i-1].o==l[i].o and l[i].o==l[i+1].o)
       or (l[i-1].e==l[i].e and l[i].e==l[i+1].e))
   then s[i]=1;
otherwise
   s[i]=-1;
```

Proof. In the first case the three lines end or start in the same vertex, so they form a triangle, as in type (a), and in the second case they form a quadrilateral, as in type (b). □

Bibliography

[AKMV05] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, Comm. Math. Phys. **254**(2), 425–478 (2005).

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Aud91] M. Audin, *The topology of torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1991, Translated from the French by the author.

[BIP10a] R. Birkner, N. O. Ilten and L. Petersen, *Computations with equivariant toric vector bundles*, The Journal of Software for Algebra and Geometry: Macaulay2 **2**, 11–14 (2010).

[BIP10b] R. Birkner, N. O. Ilten and L. Petersen, Computations with equivariant vector bundles (code), 2010.

[CK99] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1999.

[Cla08] P. Clarke, *Duality for toric Landau-Ginzburg models*, ArXiv e-prints (March 2008), 0803.0447.

[CLSar] D. A. Cox, J. B. Little and H. Schenck, *Toric Varieties*, Graduate Studies in Mathematics, Amer. Math. Soc., Providence, RI, To appear.

[Cox95] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4**(1), 17–50 (1995).

[Cox03] D. Cox, What is a toric variety?, in *Topics in algebraic geometry and geometric modeling*, volume 334 of *Contemp. Math.*, pages 203–223, Amer. Math. Soc., Providence, RI, 2003.

[Cra08] A. Craw, *Quiver representations in toric geometry*, ArXiv e-prints (July 2008), 0807.2191.

[Fin89] J. Fine, *On varieties isomorphic in codimension one to torus embeddings*, Duke Math. J. **58**(1), 79–88 (1989).

[Fin93] J. Fine, *Another proof of Sumihiro's theorem on torus embeddings*, Duke Math. J. **69**(1), 243–245 (1993).

[Ful93] W. Fulton, *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[GKM+on] E. Gasparim, T. Köppe, K. Majumdar, J. Martínez García and K. Ray, *Partition functions for Topological String Theories on Singular Varieties*, (In preparation).

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Har95] J. Harris, *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original.

[Hau03] H. Hauser, *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*, Bull. Amer. Math. Soc. (N.S.) **40**(3), 323–403 (electronic) (2003).

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79**, 205–326 (1964).

[HKK+03] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*, American Mathematical Society, Providence, RI, 2003, With a preface by Vafa.

[Hum75] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21.

[IKP06] A. Iqbal and A.-K. Kashani-Poor, *The vertex on a strip*, Adv. Theor. Math. Phys. **10**(3), 317–343 (2006).

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*, Springer-Verlag, Berlin, third edition, 1994.

[Mus94] I. M. Musson, *Differential operators on toric varieties*, J. Pure Appl. Algebra **95**(3), 303–315 (1994).

[Nag10] K. Nagao, *On higher rank Donaldson-Thomas invariants*, ArXiv e-prints (February 2010), 1002.3608.

[NY09] K. Nagao and M. Yamazaki, *The Non-commutative Topological Vertex and Wall Crossing Phenomena*, (2009), 0910.5479.

[OSY10] H. Ooguri, P. Sulkowski and M. Yamazaki, *Wall Crossing As Seen By Matrix Models*, (2010), 1005.1293.

[Sum74] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14**, 1–28 (1974).

[Sum75] H. Sumihiro, *Equivariant completion. II*, J. Math. Kyoto Univ. **15**(3), 573–605 (1975).

[Sze08] B. Szendrői, *Non-commutative Donaldson-Thomas invariants and the conifold*, Geom. Topol. **12**(2), 1171–1202 (2008).