EXERCISES ON MODULI OF VARIETIES

Some of these exercises are computational/explicit and some are not. Feel free to skip problems that are not interesting to you!

(1) If C is a smooth curve of genus $g \geq 2$, prove that either ω_C is very ample or C is hyperelliptic. If C is hyperelliptic, prove there exists an embedding $C \subset \mathbb{P}(1,1,g+1)$.

(2) (a) Let $C = (x^2 = y^3) \subset X = \mathbb{A}^3$. Compute the log canonical threshold $\operatorname{lct}(X, C)$.
(b) Do the same for $C = (x^2 = y^4)$.
(c) (*Only* if you have seen weighted blow-ups.) Do the same for $C = (x^a = y^b)$.

(3) For n, a relatively prime positive integers, let $\frac{1}{n}(1, a)$ denote the quotient singularity $\mathbb{A}^2/\mu_n$ where an nth root of unity acts on points (x, y) by $\zeta_n \cdot (x, y) = (\zeta_n x, \zeta_n^a y)$. These can be realized the varieties Spec $\mathbb{C}[x, y]^{\mu_n}$ (the ring of invariants under this action).
(a) Find a resolution of $\frac{1}{n}(1, n-1)$ and $\frac{1}{n}(1, 1)$. (If you have never seen these before: you may want to find the ring of invariants and consider these inside some affine space.) Show that they are klt.
(b) Show that $(\frac{1}{n}(1, n-1), (x = 0))$ and $(\frac{1}{n}(1, 1), (x = 0))$ are log canonical.

(4) A *smoothing* of a variety X is a flat family $\mathcal{X} \to T$ such that a closed fiber $\mathcal{X}_0 = X$ and the generic fiber $\mathcal{X}_t$ is smooth. In this lecture series, a *smoothing* is always a $\mathbb{Q}$-Gorenstein smoothing, where we assume $\mathcal{K}_{X/T}$ is $\mathbb{Q}$-Cartier.
Suppose $T = \mathbb{A}^1$ and $\mathcal{X} \to T$ is a $\mathbb{Q}$-Gorenstein family. Prove that $(-K_{\mathcal{X}_t})^{\dim X}$ is independent of $t \in T$.

(5) (a) Let $X = \mathbb{P}(a^2, b^2, c^2)$ such that $3abc = a^2 + b^2 + c^2$ where a, b, c are relatively co-prime. Prove that the singularities of X are locally smoothable (i.e., in some neighborhood of each singular point of X, we can construct a smoothing).
(b) If you know a bit of deformation theory, show that there are no local-to-global obstructions to smoothing X, and deduce that X is smoothable.
(c) Assuming (b), prove that X is smoothable to $\mathbb{P}^2$.

(6) If $\mathbb{P}(p, q, r)$ is any weighted projective degeneration of $\mathbb{P}^2$, prove that $\mathbb{P}(p, q, r)$ must be as in problem 5(a).

(7) Prove that the smooth quadric surface is K-semistable and that there exists a K-semistable cubic surface. Deduce that the generic cubic surface is K-semistable.

(8) Prove the classification of GIT-semistable quartic plane curves.
(a) Prove that a quartic with a triple point is GIT unstable (exhibit a destablizing one-parameter subgroup).
(b) Prove that the union of a cubic curve and an inflectional line is GIT unstable (exhibit a destabilizing one-parameter subgroup).
(c) Prove that the following quartic curves are GIT semistable: any curve with at worst A_n singularities other than the one in (b), and the double conic. (Reminder/definition: an A_n singularity is a curve singularity of the form $x^2 = y^{n+1}$. What is the maximal such n that can occur on a quartic curve?)

(9) Let $(X, cD) = (\mathbb{P}^2, c(xz - y^2)^2)$. If $E = (xy - z^2)$ is the conic in $\mathbb{P}^2$, compute $A_{X,cD}(E)$ and $S_{X,cD}(E)$. Using that $\delta(X, cD) = \inf_E \frac{A_{X,cD}(E)}{S_{X,cD}(E)}$, find an upper bound c_0 for which (X, cD) could be K-semistable.

(10) For $c = c_0 + \epsilon$, where c_0, (X, cD) is as in the previous problem, the pair (X, cD) is K-unstable. Properness of K-moduli tells us any family of smooth plane quartics degenerating to this pair has a K-semistable limit. Find this limit, assuming the family of smooth plane quartics is sufficiently generic.

(11) Let $c_0 = \text{lct}(\mathbb{P}^2, D)$ where D is a quartic curve with a single cusp. (You found this number in (2).) For $c = c_0 + \epsilon$, the pair $(\mathbb{P}^2, cD)$ is KSBA-unstable. Properness of KBSA-moduli tells us that any family of smooth plane quartics degenerating to this pair has a KSBA stable limit (a limit with slc singularities and ample log canonical). Find this limit, assuming the family of smooth plane quartics is sufficiently generic.

(12) We asserted that the KSB(A) moduli stack is a Deligne-Mumford stack, which necessitates that the automorphism groups are finite. Prove this: if X is a variety with ample K_X, then $\text{Aut}(X)$ is finite.

(13) Prove that the K-moduli stack cannot be Deligne-Mumford in general.

(14) Show by example that two non-isomorphic algebraic stacks can have isomorphic good moduli spaces.