EXERCISES ON GEOMETRIC INVARIANT THEORY - AGGITATE

Exercise 1. Show that the additive group $\mathbb{G}_a$ is not linearly reductive by finding a finite dimensional $\mathbb{G}_a$-representation V such that $V^{\mathbb{G}_a}$ has no $\mathbb{G}_a$-invariant complement.

Exercise 2. Let G be a finite group whose order is invertible in k. Show that G is linearly reductive.

Exercise 3. [Affine algebraic groups are linear algebraic groups]
Let G be an affine algebraic group. We will prove that G is a closed subgroup of GL_n.

a) For an action of G on an affine scheme $X = \mathrm{Spec} A$, show any finite dimensional vector space $W \subset A$ is contained in a finite dimensional G-invariant vector space $V \subset A$.
b) Apply part (a) to the action of G on itself by left multiplication to find a finite dimensional G-invariant vector space of $V \subset \mathcal{O}(G)$ containing a set of generators for $\mathcal{O}(G)$. Prove that G is a linear algebraic group by constructing a morphism $G \to \mathrm{Mat}_{n\times n}$ where $n = \dim V$ by applying the comultiplication $m^*\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ to a basis of V.

Exercise 4. [Properties of the Hilbert–Mumford weight]
For a group G acting on $X = \mathbb{P}(V)$ via a representation $G \to \mathrm{GL}(V)$, recall that the Hilbert–Mumford weight of $x = [v] \in X$ with respect to a 1-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ is

$$\mu(x, \lambda) = -\min\{\mathrm{wt}_{\lambda(\mathbb{G}_m)}(v)\}.$$

Prove the following statements.

a) The Hilbert–Mumford weight $\mu(x, \lambda)$ is the unique integer $\mu \in \mathbb{Z}$ such that $\lim_{t\to 0} t^\mu \lambda(t) \cdot v$ exists[1] and is non-zero.
b) We have $\mu(x, \lambda) = \mu(x_0, \lambda)$ where $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$ (this limit exists in X by the valuative criterion for properness).
c) We have $\mu(x, \lambda) \leq 0$ if and only if $\lim_{t\to 0} \lambda(t) \cdot v$ exists and equality holds if and only if $\lim_{t\to 0} \lambda(t) \cdot v \neq 0$.

Exercise 5. Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ by $t \cdot [x_0 : \cdots : x_n] = [tx_0 : tx_1 : \cdots : tx_{n-1} : t^{-1}x_n]$.

a) Describe the (semi)stable locus using the Hilbert–Mumford criterion and from this determine the GIT quotient.
b) Compute the invariant ring to give an alternative way to describe the GIT quotient.

Exercise 6. [Binary forms of degree d] Consider the action of SL_2 on $X = \mathbb{P}(k[x, y]_d) \cong \mathbb{P}^d$. For a degree d binary form $F(x, y) = \sum_{i=0}^d a_i x^{d-i} y^i$, show that $[F]$ is semistable if and only if any root of F has multiplicity at most $d/2$. When is F stable?

Exercise 7. [Twisted affine GIT]
For a reductive group G acting on an affine space $\mathbb{A}^n$, recall we can twist the linearisation on $\mathcal{O}_{\mathbb{A}^n}$ by a character $\rho\colon G \to \mathbb{G}_m$ to obtain a good quotient of a ρ-semistable set.

a) For $G = \mathbb{G}_m$ acting by scalar multiplication on $V = \mathbb{A}^n$, compute the ρ-(semi)stable locus and the twisted affine GIT quotient $V/\!/_\rho G$ for all characters $\rho\colon \mathbb{G}_m \to \mathbb{G}_m$ (which correspond to integers $n \in \mathbb{Z}$.
b) For $r < n$), consider $G = \mathrm{GL}_r$ acting on $\mathrm{Mat}_{r\times n}$ by left multiplication and the character $\det\colon \mathrm{GL}_r \to \mathbb{G}_m$. Show that $A \in \mathrm{Mat}_{r\times n}$ is semistable (with respect to det) if and only $\mathrm{rk}(A) = r$ by using King's Hilbert–Mumford criterion for twisted affine GIT. Then describe the twisted affine GIT quotient.

[1] Recall that we say this limit exists if the map $\mathbb{G}_m \to V$ given by $t \mapsto t^\mu \lambda(t) v$ extends to $0 \in \mathbb{A}^1$ and we write the above limit for the image of 0 in this case.