WALL CROSSING IN MODULI OF VARIETIES

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These notes were written for the 2024 AGGITATE LMS Summer School at the University of Essex. They intend to serve as an introduction to one perspective on wall-crossing in moduli of varieties. These lecutres happened simultaneously with courses on GIT, K-stability, and stacks and good moduli spaces, so are not entirely self-contained.

Contents

For what follows, we will be working over $\mathbb{C}$.

1. What is a moduli space?

For these lectures, we aim to discuss moduli spaces of varieties. Loosely, a moduli space is a *parameter* space for some type of geometric object. We want to construct a set M where each point $p \in M$ represents a particular geometric object. An optimistic hope may be to do this in such a way that:

Goal 1.1. *M is a projective variety.*

If M happens to be a variety (or at least a topological space), then it makes sense to talk about *families* of geometric objects. A curve inside M should represent a *family* of objects parameterized by that curve. If M is projective, then it makes sense to talk about 'limits' of families: for example, give a family of objects over some punctured curve $T - \{0\}$, which should correspond to a morphism $T - \{0\} \to M$, because M is proper, this extends to a morphism $T \to M$. Therefore, there is some object corresponding to $0 \in T$, and therefore we can complete our family of objects over $T - \{0\}$ to a family of objects over T. We refer to this as the 'limit' of the family of objects.

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2. Moduli of curves

Definition 2.1. A curve C is a one-dimensional connected projective variety.

2.1. M_g **the moduli space of smooth genus** g **curves.** Suppose C is a smooth curve and $g(C) \geq 2$. Then, ω_C is an ample line bundle, so some power defines an embedding

$$|\omega_C^{\otimes n}| : C \hookrightarrow \mathbb{P}^N.$$

Exercise 2.2. For C smooth with $g(C) \geq 2$, show that there is a uniform n such that $\omega_C^{\otimes n}$ is very ample for all C. (Hint: n should be small.)

Using the previous exercise, we can embed all curves of a fixed genus $g \geq 2$ into a projective space $\mathbb{P}^N$. Therefore, there is some open locus U of the Hilbert scheme parameterizing all smooth curves of genus g. If we can make sense of the quotient $U/Isom$ where we quotient by the isomorphism relation, then we have succeeded in constructing a moduli space of smooth genus g curves. Let M_g be this space:

Definition 2.3. The moduli space of smooth curves M_g is the set of all smooth projective genus $g \geq 2$ curves up to isomorphism.

By the outlined construction above, M_g turns out to be a quasi-projective variety. However, M_g is **not** proper: we can take families of smooth curves and degenerate them to singular curves! How do we compactify? We wish to find some proper (projective?) variety $\overline{M}$ such that $M_g \subset \overline{M}$. But, we don't want an arbitary $\overline{M}$: we would like $\overline{M}$ to be a moduli space, i.e. we would like the points of $\overline{M}$ to correspond to the varieties that arise as limits of families in M_g. This is often called a *modular* compactification of M_g. There is a 'standard' way to do this, but there is **not** a unique way to do this. There are many modular compactifications of M_g! We will discuss some throughout the lecture series. First, we construct the usual one.

2.2. $\overline{M}_g$ **the moduli space of stable genus** g **curves.** In our sketch of a construction of M_g above, we restricted to the locus of the Hilbert scheme parameterizing smooth curves. We could attempt to compactify M_g by looking at a larger class of curves that is 'self-contained', i.e. already contains all of the limits. The usual class of curves we consider are called **stable** curves.

Definition 2.4. A curve C of genus $g \geq 2$ is **stable** if:

1. C has at worst nodes as singularities, i.e. the points of C are either smooth or locally isomorphic to $xy = 0$, and
2. the canonical sheaf ω_C is ample. As C is connected, this means that any component $C_i \subset C$ that is isomorphic to $\mathbb{P}^1$ meets the rest of the curve in at least three points.

Exercise 2.5. Prove that this last condition is equivalent to finiteness of $\mathrm{Aut}(C)$ (which is often given in place of (2)).

Exercise 2.6. For a stable curve C, $\omega_C^{\otimes 3}$ is very ample.

Now, we can proceed in the same way as before: use a power of the canonical line bundle to embed all stable curves of a fixed genus into a projective space $\mathbb{P}^N$, and consider the subspace of the Hilbert scheme parameterizing such curves. Taking the quotient by isomorphism (again, we are skipping precisely what this means and any proofs of local closedness of the locus of stable curves in the Hilbert scheme) yields the space $\overline{M}_g$.

Definition 2.7. The moduli space $\overline{M}_g$ is the set of all stable genus $g \geq 2$ curves up to isomorphism.

Because we have enlarged M_g to $\overline{M}_g$, we in fact get a proper variety.

Theorem 2.8 (Deligne-Mumford, 1969)**.** *$\overline{M}_g$ is proper.*

The concrete consequence is:

any family of smooth curves over some punctured base can be completed (uniquely!) to a family of stable curves.

2.3. **Properness of $\overline{M}_g$.** The first thing we will do is sketch the properness of $\overline{M}_g$.

Theorem 2.9. *For $g \geq 2$, $\overline{M}_g$ is proper.*

Proof. (Sketch.) We approach the proof using the valuative criterion for properness. Let $T = \operatorname{Spec} R$ be a DVR and suppose $T - \{0\} \to \overline{M}_g$ is a map with generic point in M_g. Geometrically, this gives us a smooth family of curves $S \to T - \{0\}$ (really: we may need to take a base change to get this to exist and/or start by working with the moduli stack instead of the moduli space). Let $\overline{S}$ be any flat completion of the family to $\overline{S} \to T$. This adds something as the fiber over $\{0\}$ which is some curve that may be unstable. Because we are working over $\mathbb{C}$, regardless of what we added over $\{0\}$, we can resolve the singularities.

Up to base change, blow-up the central fiber repeated to get a family $\tilde{S} \to T'$, where the central fiber $\tilde{S}_0$ is reduced and normal crossing. This is almost a stable curve: it has at worst nodes at every point by construction. To make it stable, we need to remove any pieces where $\omega_{\tilde{S}_0}$ is not ample.

As an exercise: show that the only components of $\tilde{S}_0$ for which $\omega_{\tilde{S}_0}$ is not ample are $\mathbb{P}^1$s meeting the rest of the curve at one point (which are -1 curves in $\tilde{S}_0$) or $\mathbb{P}^1$s meeting the rest of the curve at two points (which are -2 curves). Classical birational geometry tells us we can contract these components: the resulting surface is the canonical model of S, and yields a completion of the original family with stable central fiber. □

2.4. **Other compactifications of M_g.** There are many other compactifications of M_g, and one goal of this lecture series will be to study how these compactifications are related.

Definition 2.10. A curve C of genus $g \geq 2$ is **pseudostable** if:

(1) Every point $p \in C$ is either smooth, nodal (locally $xy = 0$), or cuspidal (locally, $x^2 = y^3$).
(2) C has no *elliptical tails*, meaning C cannot be decomposed as $C = C' \cup E$ where $g(E) = 1$ and $E \cap C'$ consists of just one point.
(3) ω_C is ample.

Similar to the case of $\overline{M}_g$, one can show that there is a proper moduli space $\overline{M}_g^{ps}$ parameterizing all pseudostable curves.

Remark 2.11. If C is a stable curve, then C is pseudostable if and only if C does not contain an elliptic tail. If C is pseudostable, then C is stable if and only if C has no cusps. Therefore, $\overline{M}_g$ is birational to $\overline{M}_g^{ps}$, but they are **not** the same.

Because these two varieties are birational, there is a rational map

$$\overline{M}_g \dashrightarrow \overline{M}_g^{ps}$$

and one could ask if there is a *modular* way to resolve it. In this case, it turns out that there is in fact a morphism

$$\overline{M}_g \to \overline{M}_g^{ps}$$

that contracts the locus of curves with elliptic tails to the locus of curves with cusps. We can even study this contraction on the curves themselves: given a family of curves degenerating to a curve with an elliptic tail, we can contract the elliptic tail in the family to see that the family of curves degenerates to a cuspidal curve. In other words, we get a *different* limit of the family of curves in the two moduli spaces. In fact, this contraction is an instance of the *minimal model program* on $\overline{M}_g$.

To see another example, we'll focus on a specific genus. Let $g = 3$.

Exercise 2.12. If $g = 3$, the generic genus 3 curve can be written as a quartic plane curve in $\mathbb{P}^2$.

Therefore, generically, genus 3 curves are sections of $\mathcal{O}(4)$ on $\mathbb{P}^2$, so $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4)))$ is a parameter space for genus 3 curves and we can find its GIT quotient $\overline{M}^{GIT}$. As any smooth nonhyperelliptic genus 3 curve is parameterized $\overline{M}^{GIT}$, it is birational to $\overline{M}_g$, so again we have

$$\overline{M}_g \dashrightarrow \overline{M}^{GIT}.$$

These are not the same: there are **no** hyperelliptic curves parameterized by $\overline{M}^{GIT}$, because hyperelliptic genus 3 curves cannot be written as plane curves–the canonical map is a double cover. However, we can resolve this rational map in a modular way. We'll get back to this later in the week, but for now, Figure 1 is a picture of the resolution of the map. There is a special point $p \in \overline{M}^{GIT}$ that parameterizes the double conic $(xy - z^2)^2 = 0$ (which, morally, is the 'image' of the hyperelliptic locus) that we can blow-up. This turns out to give a new moduli space, and the exceptional divisor of this blow-up parametrizes all of the hyperelliptic curves. The rest of the steps are illustrated in Figure 1, and the labels will make more sense after the rest of the lectures when the same picture will return.

3. KSB(A) moduli of canonically polarized varieties and pairs

The higher dimensional analogue of $\overline{M}_g$ is given by the following theorem:

Theorem 3.1. *Fix a positive integer n, a positive rational number V, and a nonnegative rational number b. Then, there exists a Deligne-Mumford stack $\mathcal{M}^{KSB(A)}_{n,V,b}$ and a projective coarse moduli space $M^{KSB(A)}_{n,V,b}$ parametrizing slc pairs (X, D) such that $\dim X = n$, $(K_X + D)^n = V$, $D = b\Delta$ for some effective $\mathbb{Z}$-divisor Δ, and $K_X + D$ is ample.*[1]

To make sense of this theorem, we need to define 'slc.'

3.1. Singularities.

Definition 3.2. A variety X is $\mathbb{Q}$-Gorenstein if K_X is a $\mathbb{Q}$-Cartier divisor, i.e. some positive multiple mK_X is Cartier.

Definition 3.3. If X is a normal $\mathbb{Q}$-Gorenstein variety, let $\pi : Y \to X$ be a resolution of singularities. Let $\{E_i\}$ be the divisorial components of the exceptional locus. As K_X is $\mathbb{Q}$ Cartier and K_Y and π^*K_X agree outside of $\operatorname{Supp} \cup_i E_i$, there exist rational numbers $a_i \in \mathbb{Q}$ such that

$$K_Y = \pi^* K_X + \sum a_X(E_i) E_i.$$

The discrepancy of X is

$$\operatorname{discrep}(X) = \inf_{E/X : E \text{ exceptional}} a_X(E).$$

The infimum ranges over all exceptional divisors in any resolution of X.

The variety X is said to have:

- *terminal singularities* if $\operatorname{discrep}(X) > 0$,
- *canonical singularities* if $\operatorname{discrep}(X) \geq 0$,
- *log terminal singularities* if $\operatorname{discrep}(X) > -1$, and
- *log canonical singularities* if $\operatorname{discrep}(X) \geq -1$.

[1] See [Kol23] for more detail. There, several different moduli functors are presented and we use KSB in the case $b \in \{0, \{1 - \frac{1}{m}\}\}$ and KSBA for arbitrary b.

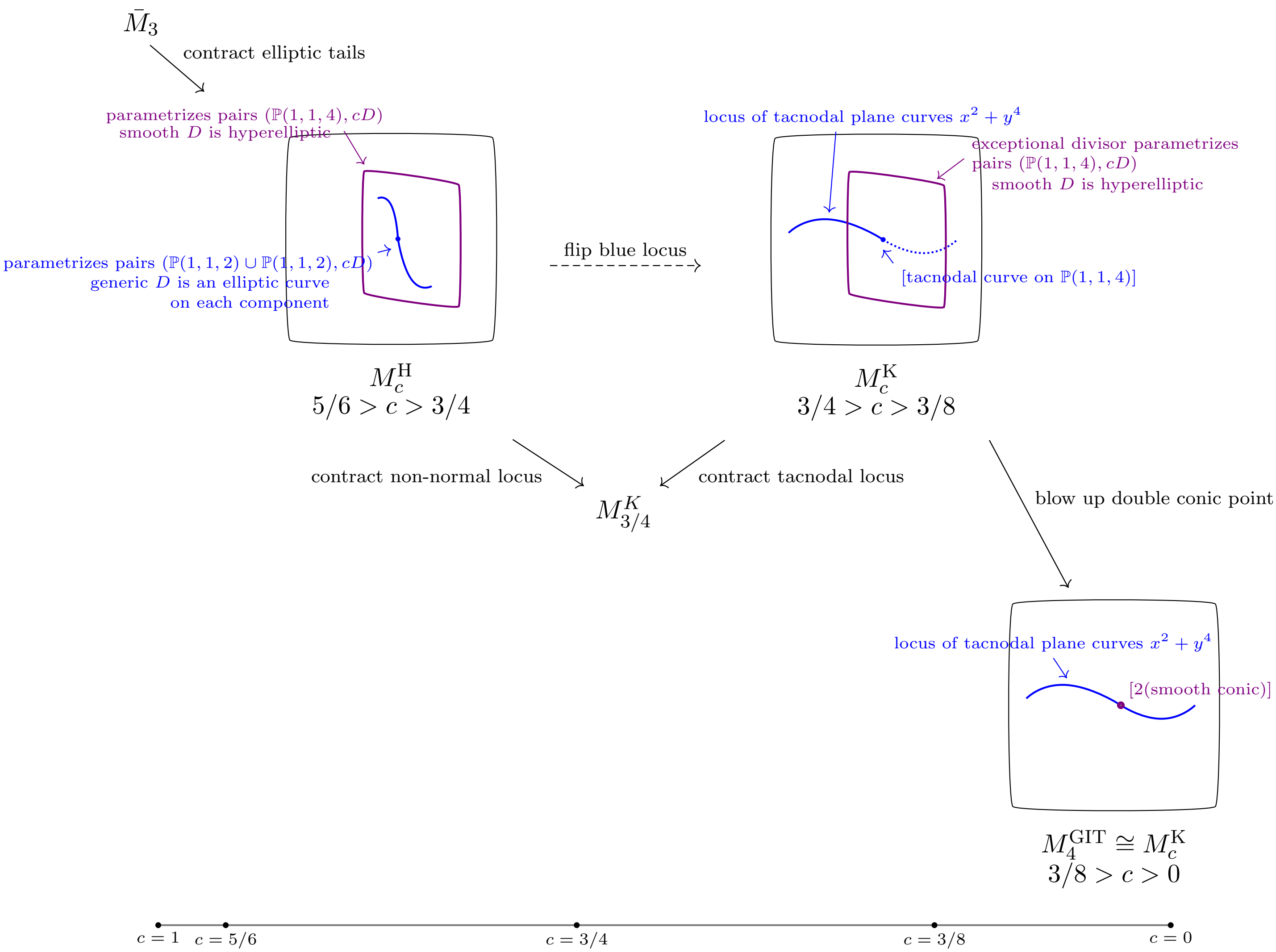

FIGURE 1. Wall crossings for moduli of quartic curves.

While the definition of singularities was given in terms of discrep(X), which ranges over all divisors over X, the following is true:

Theorem 3.4. *Let X be a $\mathbb{Q}$-Gorenstein normal variety. Let $\pi : Y \to X$ be a resolution with exceptional divisors $\{E_i\}$ and write $K_Y = \pi^* K_X + \sum a_X(E_i) E_i$. Then,*

$$\operatorname{discrep}(X) = \min\{\min_i \{a_X(E_i)\}, 1\}.$$

In other words, X has terminal singularities if $a_X(E_i) > 0$ for all i, canonical singularities if $a_X(E_i) \geq 0$ for all i, log terminal singularities if $a_X(E_i) > -1$ for all i, and log canonical singularities if $a_X(E_i) \geq -1$ for all i.

In particular, it suffices to check the inequalities on a single resolution of X.

Example 3.5. Let X be the cone $xy = z^2 \subset \mathbb{A}^3$. Blowing up the origin $(0,0,0) \subset \mathbb{A}^3$, we obtain $\pi : Z \to \mathbb{A}^3$. Let Y be the strict transform of X and let $\pi_Y = \pi|_Y : Y \to X$. This is a resolution of singularities of X. Indeed, Y is smooth by the local equations of the blow-up. Furthermore, if

$E \subset Z$ is the exceptional divisor of π, $E \cong \mathbb{P}^2$, and $E_Y := E \cap Y$ is the exceptional divisor of π_Y. Then, E_Y is a conic in E (seen from the blow up equation) and $(E_Y)^2 = E \cdot E \cdot Y = E|_E \cdot Y|_E$. Because $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^2}(-1)$, we see that $(E_Y)^2 = -2$.

Therefore, we may compute $a_X(E_Y)$: consider the equality

$$K_Y = \pi_Y^* K_X + a_X(E_Y) E_Y.$$

Adding E_Y to each side, we obtain

$$K_Y + E_Y = \pi_Y^* K_X + (a_X(E_Y) + 1) E_Y.$$

Taking the intersection with E_Y, we find

$$\begin{aligned}(K_Y + E_Y) \cdot E_Y &= (\pi_Y^* K_X + (a_X(E_Y) + 1) E_Y) \cdot E_Y \\ -2 &= 0 + (a_X(E_Y) + 1)(-2)\end{aligned}$$

where the left side holds by adjunction and $\pi_Y^* K_X \cdot E_Y = 0$ by the projection formula. Therefore, $a_X(E_Y) + 1 = 1$, so $a_X(E_Y) = 0$. In particular, we have found $\mathrm{discrep}(X) = 0$ and hence X is canonical.

Remark 3.6. Canonical surface singularities, like the previous example, are often called du Val singularities or ADE singularities. The 'ADE' refers to the Dynkin diagram that is the dual graph of the minimal resolution.

We can also define singularities of a pair (X, D), where D is a divisor on X, using a log resolution and a similar definition.

Definition 3.7. If (X, D) is a log pair with X normal and $K_X + D$ $\mathbb{Q}$-Cartier, write $D = \sum b_j D_j$ where $\{D_j\}$ are prime divisors. Let $\pi : Y \to X$ be a log resolution of singularities and D_Y the strict transform of D on Y. Let $\{E_i\}$ be the components of the exceptional locus. Then, for some rational numbers $a_{X,D}(E_i)$, we can write

$$K_Y + D_Y = \pi^*(K_X + D) + \sum_{E_i} a_{X,D}(E_i) E_i.$$

As above, we define the discrepancy of the pair (X, D) to be

$$\mathrm{discrep}(X, D) = \inf_{E/X : E \text{ exceptional}} a_{X,D}(E).$$

Note the infimum is taken over all exceptional divisors in any log resolution of (X, D).

The pair (X, D) is said to have:

- *terminal singularities* if $\mathrm{discrep}(X, D) > 0$ and $b_j < 1$ for all j,
- *canonical singularities* if $\mathrm{discrep}(X, D) \geq 0$,
- *kawamata log terminal singularities (klt)* if $\mathrm{discrep}(X, D) > -1$ and $b_j < 1$ for all j, and
- *log canonical singularities* if $\mathrm{discrep}(X, D) \geq -1$.

[2]

As in the previous case, we can check these on one log resolution, provided we resolve so that $\mathit{Supp} D_Y$ is smooth and $\mathit{Supp}(D_Y \cup (\cup E_i))$ is normal crossing (i.e. any intersections among the D_j are separated, and the strict transforms of the D_j meet the exceptional divisors transversally).

Finally, we will need to understand non-normal analogues of these singularities. Roughly, this is equivalent to requiring that the normalization satisfies the notions above. We only define semi-log-canonical.

[2] If D_j is one of the components of the divisor, we define the discrepancy of D_j to be $a_{X,D}(D_j) = -b_j$. This will be important later.

Definition 3.8. Suppose X is a demi-normal[3] projective variety with $\mathbb{Q}$-Cartier canonical divisor K_X. Then, X is **semi-log-canonical (slc)** if and only if (X^ν, Δ^ν) is log canonical, where $\nu : X^\nu \to X$ is the normalization and Δ^ν is the conductor.

Exercise 3.9. If $\dim X = 1$, prove that slc is equivalent to nodal, and hence the KSB moduli space for 1-dimensional varieities is equivalent to $\overline{M}_g$.

Remark 3.10. Unlike the case of curves, it is typically very hard to describe any KSB(A) moduli space explicitly. Even for surfaces, where there exists a complete classification of slc singularities, we typically do not have an understanding of these spaces. Part of the difficulty is in the deformation theory: even on a surface, if a given slc singularity happens to be locally smoothable, a surface with that singularity need not be globally smoothable.

4. K-stability and moduli of Fano varieties

Next, we study moduli of Fano varieties. It is more subtle than the previous notions for canonically polarized varieties. We need to invoke a stronger stability condition to get well-behaved moduli of spaces. The main theorem is:

Theorem 4.1. *Fix a positive integer n, a positive rational number V, and a nonnegative rational number b. There is an Artin stack of finite type $\mathcal{M}^K_{n,V,b}$ and a projective good moduli space $M^K_{n,V,b}$ parameterizing K-semistable and K-polystable, respectively, pairs (X, D) with $\dim X = n$, $(-(K_X + D))^n = V$, $D = b\Delta$ for some effective $\mathbb{Z}$-divisor Δ, and $K_X + D$ antiample.*

So, we next try to answer: what is K-stability?

4.1. Introduction to K-stability.

Our first goal is to introduce the notion of K-stability with connections to other invariants. We focus on K-moduli of Fano varieties X, but all of the definitions can be extended to K-moduli of log Fano pairs (X, D).

4.1.1. *Fano varieties and history of K-stability.*

Definition 4.2. A smooth variety X is called a **Fano** variety if $-K_X$ is ample (i.e. for some $m \gg 0$, the rational map $|-mK_X| : X \dashrightarrow \mathbb{P}(H^0(-mK_X))$ is an embedding).

If $\dim X = 1$, X is Fano if and only if $X = \mathbb{P}^1$. In general, $\mathbb{P}^n$ is Fano for any n, but there are many other types of Fano varieties in higher dimensions. When $\dim X = 2$, a Fano surface is called a *del Pezzo* surface.

Before we define K-stability, we detour into its historical origins.

It is an old(er) question in differential geometry to study when Fano varieties can be equipped with a Kähler-Einstein (KE) metric. We won't be using this perspective in this series of lectures, but it provides relevant background on how K-stability came to be.

A smooth Kähler variety with Kähler form ω is said to have a KE metric if ω satisfies the Einstein equation

$$\mathrm{Ric}(\omega) = \lambda\omega$$

for some constant λ. In the typical trichotomy of varieties–K_X ample, trivial, or antiample–a smooth projective variety with ample canonical class always admits a KE metric, proved independently by Aubin and Yau in 1978, and one with trivial canonical class always admits a KE metric, proved by Yau. However, for Fano varieties, it was known much earlier that they cannot always admit a KE metric. For example, in 1957, Matsushima proved that if X is KE, then $\mathrm{Aut}(X)$ is reductive. Therefore, it was of interest to differential geometers to formulate a notion for Fano varieties that precisely captured the existence of a KE metric.

[3]A variety is **demi-normal** if it is S_2 and is smooth or nodal in codimension 1.

Several years later, the notion of K-stability was introduced. In 1992, Ding and Tian introduced the generalized Futaki invariant to capture the existence of a KE metric, and proved that the existence of such a metric implies this invariant is non-negative. In 1997, Tian (analytically) and later Donaldson in 2002 (algebraically), the notion of K-stability was formally defined using the Futaki invariant, and the *Yau-Tian-Donaldson Conjecture* was made: a smooth Fano variety is K-polystable if and only if it admits a KE metric. This conjecture was proven by Chen, Donaldson, and Sun in 2012, and has since been extended beyond the smooth case. We will define K-stability below, but you may be wondering:

Question 4.3. What does this have to do with algebraic geometry?

As we'll see shortly, the algebraic formulation of K-stability looks like other powerful notions in algebraic geometry (for example, GIT), so +1 for motivation to study it. Also, it has something to do with degenerating varieties in families, so +1 for connecting to moduli problems. However, it is *remarkable* that this differential geometric notion is exactly the correct thing to study to get well-behaved moduli spaces of Fano varieties, and *remarkable* that it has so many connections to older algebro-geometric concepts (e.g. singularities and the minimal model program).

In the words of Chenyang Xu, *"The concept of K-stability is one of the most precious gifts differential geometers brought to algebraic geometers."*

4.2. **K-stability via test configurations.** Without further ado, let's define K-stability (which we do only for Fano varieties). We will not restrict ourselves to the smooth world; let us consider normal projective varieties. The original definition is due to Tian and Donaldson but we give an algebraic refinement below.

Definition 4.4. Let $(X, -K_X)$ be a polarized projective variety of dimension n, and suppose X is normal. Because $-K_X$ is ample, for $m \gg 0$, there is an embedding $|-mK_X| : X \to \mathbb{P}^N$. For any action $\mathbb{G}_m$ on PGL_{N+1}, there in an induced action $\mathbb{G}_m$ on the class $[X] \in \mathrm{Hilb}(\mathbb{P}^N)$. Let $[X_0] = \lim_{t\to 0} t \cdot [X]$.

A **special test configuration** is the induced family

$$\begin{array}{ccc} \mathbb{G}_m \times (X, -mK_X) & \longrightarrow & (\mathcal{X}, \mathcal{L}) \\ \downarrow & & \downarrow \\ \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} & \longrightarrow & \mathbb{A}^1 \end{array}$$

where X_0 is assumed to be a klt Fano variety.

Given a special test configuration, we can 'compactify': complete the family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{A}^1$ to a family $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ over $\mathbb{P}^1$ by adding the trivial fiber $(X, -mK_X)$ over $\infty \in \mathbb{P}^1$. We do this by gluing the family $(\mathcal{X}, \mathcal{L})$ to the trivial family $X \times \mathbb{P}^1 \setminus \infty$ along $\mathbb{A}^1 \setminus 0$.

Definition 4.5. The **Futaki invariant** $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ of the test configuration $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -\frac{1}{2(-K_X)^n(n+1)} \left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1}\right)^{n+1}.$$

Definition 4.6 (Tian, Donaldson)**.** Let X be a variety such that $-K_X$ is ample. X is

(1) K-semistable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$.
(2) K-stable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$, and equality holds if and only if $(\mathcal{X}, \mathcal{L})$ is trivial.
(3) K-polystable if X is K-semistable and, if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$, then $\mathcal{X} \cong X \times \mathbb{A}^1$.

Remark 4.7. If you are familiar with GIT, think of 'polystable' as a closed orbit condition as it is in GIT.

Remark 4.8 (Red Flag!)**.** To test if a variety is K-(semi/poly)stable, we must a priori test *infinitely* many test configurations, which depend on the $\mathbb{G}_m$ action *and* the power m used in the embedding $|-mK_X| : X \to \mathbb{P}^N$. (For those familiar with GIT: this is like checking the Hilbert-Mumford weight for every possible embedding of X into a higher and higher projective space.) How can this possibly be reasonable?

We can begin to simplify this making connections to other quantities in algebraic geometry. First, although nothing about singularities explicitly appears in the test configuration definition, asking that a variety is K-semistable has (surprising!) consequences on the singularities of X. For example:

Theorem 4.9 (Odaka)**.** *If X is normal and $-K_X$ is ample, then K-semistability of X implies that X has log terminal singularities.*

4.3. K-stability via the δ invariant.

Definition 4.10. Let X be a $\mathbb{Q}$-Fano variety and E a prime divisor over X. Let $\mu : Y \to X$ be any morphism such that $E \subset Y$.

Let $A_X(E)$ be the log discrepancy of the divisor E, or the number

$$A_X(E) = 1 + \mathrm{ord}_E(K_Z - f^*K_X) = 1 + a_X(E).$$

Define $S_X(E)$ to be

$$S_X(E) = \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

This does not depend on choice of μ and Y, so we often write

$$S_X(E) = \frac{1}{(-K_X)^n}\int_0^\infty \mathrm{vol}(-K_X - tE)dt.$$

The δ-invariant of E is

$$\delta_X E = \frac{A_X(E)}{S_X(E)}.$$

In the integral S, we must compute $\mathrm{vol}(D)$ for $D = -K_X - tE$. What follows are some notions related to volumes of divisors.

Definition 4.11. The volume a divisor D on a normal variety X of dimension n is

$$\mathrm{vol}(D) = \lim_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Definition 4.12. A divisor D on a normal variety X of dimension n is **big** if one of the following equivalent definitions hold:

(1) For $m \gg 0$, the map given by the linear system $|mD| : X \dashrightarrow \mathbb{P}^N$ is birational onto its image.
(2) For $m \gg 0$, there exists a constant $c > 0$ such that $h^0(X, mD) > cm^n$.
(3) $\mathrm{vol}(D) > 0$.

How do we compute volumes? If D is a divisor on a normal variety X of dimension n, we have the following:

(1) If D is nef, $\mathrm{vol}(D) = D^n$.

(2) If D is big, by definition $\mathrm{vol}(D) > 0$, and we can at least bound the volume of D from below by considering the image of the linear system $|mD| : X \dashrightarrow \mathbb{P}^N$. Because D is big, the image is a variety Y birational to X. If we assume this a morphism $f : X \to Y$, some divisors in X may be contracted. If we write $D = f^*\mathcal{O}(1) + N$ for some effective divisor N supported on the contracted locus, then $\mathrm{vol}(f^*(\mathcal{O}(1)) \leq \mathrm{vol}(D)$ (because $h^0(f^*\mathcal{O}(1)) \subset h^0(D)$). And, $f^*\mathcal{O}(1)$ is nef, so its volume is just $\mathcal{O}(1)^n$. Therefore, we compute a lower bound for the volume of D by the volume of $\mathcal{O}(1)$. In practice, we determine N by considering MMP-like birational modifications of X. If the divisor D is trivial or negative on some effective curve in $\overline{NE(X)}$, then we contract the class of this curve. If this is divisorial, N will be supported on the divisor, and if it is a small contraction, we flip or flop the class of the curve to a new variety $X \dashrightarrow X^+$ and pullback everything to a common partial resolution $\widehat{X}$. The divisor N will be supported on the exceptional divisors of $\widehat{X} \to X$.

For context, for surfaces, this is called a *Zariski decomposition*: we can always write a big divisor D on a surface X as $D = P + N$, where P is nef, N is negative (meaning it has negative definite intersection matrix/is contractible, or is 0), and $P \cdot N = 0$. In this case, we have the equality $\mathrm{vol}(D) = \mathrm{vol}(P) = P^2$. To relate this to the birational modifications above, because N is negative, it corresponds to a contractible curve in the surface, and P is the resulting divisor on the contraction.

Remark 4.13. In the definition of $S_X(E)$, we need to compute an improper integral. However, $\mathrm{vol}(-\mu^*K_X - tE) > 0$ if and only if $-\mu^*K_X - tE$ is big. In terms of divisors on Y, the closure of the big cone of divisors is the pseudo-effective cone, so the volume is only non-zero if $t \in [0, \tau]$ where τ is the pseudo-effective threshold. This is finite; at some point we have subtracted 'too much' E and the divisor is no longer pseudo-effective. Therefore, we could re-write

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\tau \mathrm{vol}(-K_X - tE)dt.$$

With this definition, we can relate the K-(semi/poly)stability of X *intrinsically* to the birational geometry of X! The following theorem is usually called the **valuative criterion** for K-(semi/poly)stability. Initially given by Fujita and Li, there are several important contributions necessary connecting uniform K-stability and K-stability by other authors.

Theorem 4.14. *A variety X is K-semistable (resp. stable) if and only if $\delta_X(E) \geq 1$ (resp. > 1) for all prime divisors E/X.*

We provide a sample computation below.

Example 4.15. Let $X = \mathbb{F}_1$ be the blow-up of $\mathbb{P}^2$ at one point. Let's compute $\beta_X(E)$ where E is the exceptional divisor of the blow up of $\mathbb{P}^2$, the unique -1 curve on X. Because E already lives on X, we can take $\mu : X \to X$ the map extracting E to be the identity. Since $K_X = \mu^*(K_X)$, we have $A_X(E) = 1 = 1$.

Now we need to compute $S_X(E)$. We know $(-K_X)^2 = 8$. To compute $\mathrm{vol}(-K_X - tE)$, we need to understand when the divisor is ample, big, and nef. The volume is non-zero when the divisor is big (by definition of being big!). The Mori cone of X is generated by E and the class of a fiber F of the ruled surface $X \to \mathbb{P}^1$. Because $(-K_X - tE) \cdot E = 1 - (-t) = 1 + t$, this is positive on E for $t \geq 0$. Similarly, $(-K_X - tE) \cdot F = 2 - t$, so this is positive on F for $t < 2$. Because this is ample exactly when it has positive intersection with both F and E (Kleiman's criterion), this is ample for $0 < t < 2$. Also, when $t = 2$, this is trivial on F, and the morphism induced by the linear system $|m(-K_X - tE)|$ therefore contracts F and hence contracts X to a curve. Because this is not birational for any $m > 0$, the divisor is not big for any $t \geq 2$.

This implies that:

$$\text{for } 0 \le t \le 2, \quad \text{vol}(-K_X - tE) = (-K_X - tE)^2 = 8 - 2t - t^2$$
$$\text{for } t \ge 2, \quad \text{vol}(-K_X - tE) = 0.$$

So, we can compute $S_X(E)$:

$$S_X(E) = \frac{1}{8}\int_0^2 (8 - 2t - t^2)dt = \frac{7}{6}.$$

Finally, we can conclude that $\delta(E) = 6/7$, which shows that $X = \mathbb{F}_1$ is K-unstable. In particular, not ever smooth Fano is K-semistable, so we can't use K-moduli to get moduli spaces of all Fanos.

There is also a G-invariant version of the δ-invariant, which can be incredibly useful.

Definition 4.16. Suppose G is a reductive group acting on X. We define

$$\delta_G(X) = \inf_{E/X \quad G-\text{invariant}} \frac{A_X(E)}{S_X(E)}.$$

Theorem 4.17 (Zhuang)**.** *A variety X is K-semistable if and only if $\delta_G(X) \ge 1$.*

Example 4.18. If $X = \mathbb{P}^n$ and $G = PGL_{n+1}$, there are no G-invariant divisors over X, so $\delta_G(X) = \infty$ and hence X is K-semistable (in fact, K-polystable).

5. Wall crossing for moduli spaces

In the last section, we will study wall-crossing phenomena for moduli spaces. Generally, it is interesting to study moduli as some coefficient is varying. With moduli of varieties, that often means we're looking at pairs (X, cD) and allowing the coefficient c to vary.

Example 5.1. Let's consider quartic curves in $\mathbb{P}^2$ as a motivating example. A quartic curve has genus 3 and any non-hyperelliptic genus 3 curve embeds as a quartic in $\mathbb{P}^2$.

Consider a compactification of moduli of pairs $(\mathbb{P}^2, cD)$, where D is a quartic plane curve. If $0 < c < \frac{3}{4}$, this is a log Fano pair, so we can construct a K-moduli space $\mathcal{M}_c$ of pairs for each $c \in \mathbb{Q}$. If $c > \frac{3}{4}$, then this is a canonically polarized pair, so we can construct a KSBA moduli space $\mathcal{M}_c$ for each $c \in \mathbb{Q}$.

Question 5.2. How do the moduli spaces change as c varies in the interval $(0, 1)$?

In case of hypersurfaces in $\mathbb{P}^n$, we can at least establish good behavior for $c = \epsilon \ll 1$.

5.1. Relationship between K-stability and GIT.

Theorem 5.3. *For $c \ll 1$, the K-moduli stack/space parameterizing K-semi/polystable limits of pairs $(\mathbb{P}^n, cD)$, where D is a degree d hypersurface, is isomorphic to the GIT moduli stack/space of the hypersurfaces.*

First, a sketch of the proof, and then some applications.

Proof. Step 1: for any c, we show that K-semistability implies GIT semistability. Let $(\mathbb{P}^n, cD)$ be a K-semistable pair. To show that D is GIT semistable, consider a one-parameter subgroup λ. By definition, λ induces a test configuration of the pair $(\mathbb{P}^n, cD)$ (with limit over $t = 0 \in \mathbb{A}^1$ equal to $(\mathbb{P}^n, c\lim_{t\to 0}\lambda(t)\cdot D)$. By computation, the Futaki invariant of this test configuration is equal to a positive multiple of the Hilbert Mumford weight. Because $(\mathbb{P}^n, cD)$ was assumed to be K-semistable, the Futaki invariant is nonnegative and hence the Hilbert Mumford weight of λ is nonnegative and therefore D is GIT semistable.

Step 2: for $c = \epsilon \ll 1$, we show that any point $[(X, \epsilon D_X)]$ in the K-moduli space has $X = \mathbb{P}^n$ and hence D_X a hypersurface of degree d. Because $(X, \epsilon D_X)$ is K-semistable and $\epsilon \ll 1$, then linearity of the Futaki invariant with respect to the coefficient of D_X implies that X itself is K-semistable. By assumption, X is a degeneration of $\mathbb{P}^n$, but $\mathbb{P}^n$ is K-polystable, so any K-semistable degeneration of $\mathbb{P}^n$ must in fact be $\mathbb{P}^n$. Therefore, $X = \mathbb{P}^n$.

Step 3: From (1) and (2) (and openness of K-semistability) we get an open immersion $\mathcal{M}_\epsilon^K \to \mathcal{M}^{GIT}$, where the first space is the K-moduli stack and the second the GIT moduli stack. To prove this is an isomorphism, the key step is to show that it is bijective on closed points, which are precisely the polystable points. With a similar argument to (1), we can show K-polystability implies GIT polystability. Then, we must show GIT polystability implies K polystability. Suppose D_0 is GIT polystable and let $\mathcal{D} \to T$ be a family with smooth generic fiber and D_0 the special fiber. Then, the family $(\mathbb{P}^n_T, \epsilon\mathcal{D})$ has generic fiber $(\mathbb{P}^n, \epsilon D_t)$ with a smooth divisor, which is K-polystable. By properness of K-moduli, there is a K-polystable limit of this family $(X, \epsilon D_{X,0})$. If we can show $(X, \epsilon D_{X,0}) = (\mathbb{P}^n, \epsilon D_0)$, we are done. By (2), we know $X = \mathbb{P}^n$. Then, by (1), we know K-polystability of $(\mathbb{P}^n, \epsilon D_{X,0})$ implies GIT-polystability of $D_{X,0}$, so $D_{X,0}$ and D_0 are both GIT polystable limits of the same family of divisors. By separatness of the GIT moduli space, this implies $D_0 = D_{X,0}$, as desired. □

Now, a few applications of the previous theorem: because this statement holds for any $\epsilon \ll 1$, the K-moduli stacks/spaces we get are independent of ϵ sufficiently small, and therefore it makes sense to vary the coefficient and relate the moduli spaces.

Also, for low degree curves in $\mathbb{P}^2$, the GIT moduli space is well described. Therefore, this theorem says that the to completely understand the K-moduli spaces as coefficient increases, we can start with the GIT moduli space, increase the coefficient c until something "destabilizes" (which will give a "wall crossing"), find the K-semistable replacement, and continue.

Finally, this theorem also shows that K-stability informs GIT. To state an application, we recall a few results:

Proposition 5.4. *Suppose X is a Fano variety with effective divisors D, Δ such that $D \sim_{\mathbb{Q}} -rK_X$ and $\Delta \sim_{\mathbb{Q}} -r'K_X$ for rational numbers r, r'. If (X, D) and (X, Δ) are K-semistable with $-(K_X+D)$ ample and $-(K_X + \Delta)$ nef*[4]*, then $(X, tD + (1-t)\Delta)$ is K-semistable for $t \in [0, 1]$.*

This is known as *interpolation.* It can be applied readily here because:

Theorem 5.5 (Odaka)**.** *If (X, Δ) is an slc pair with $K_X + \Delta = 0$, then (X, Δ) is K-semistable.*

Therefore:

Corollary 5.6. Suppose D is a divisor on $\mathbb{P}^n$ of degree $d \geq n + 1$. If $\operatorname{lct}(\mathbb{P}^n, D) \geq \frac{n+1}{d}$, then D is GIT semistable.

Proof. By the lct assumption, if $\Delta = \frac{n+1}{d}D$, then $(\mathbb{P}^n, \Delta)$ is slc and hence K-semistable with $-(K_{\mathbb{P}^n}+\Delta)$ nef. Furthermore, $\mathbb{P}^n$ is K-semistable with $-K_{\mathbb{P}^n}$ ample. By interpolation, $(\mathbb{P}^n, (1-t)\Delta)$ is K-semistable for all $t \in [0, 1]$. In particular, if t is sufficiently close to 1, we see that $(\mathbb{P}^n, \epsilon D)$ is K-semistable and hence D is GIT semistable. □

Let's unpack these results in the case of degree 4 plane curves. We're considering moduli of pairs $(\mathbb{P}^2, cD)$ where $0 < c < 1$, D is a degree 4 curve, and trying to understand all K-semistable pairs of this form or KSBA stable pairs of this form (and their degenerations).

We know: for $c \ll 1$, the K-moduli space of pairs $(\mathbb{P}^2, cD)$ and their limits is isomorphic to the GIT moduli space. In other words, for $c \ll 1$ we start with GIT of quartic plane curves.

[4]This requires defining K-stability in a more general framework, but one can do it.

In the GIT moduli space, there is a "special" point corresponding to the double conic. This point is special as it has the largest stabilizer group out of all GIT polystable points, and is the only point in the GIT moduli space for which the pair $(\mathbb{P}^2, cD)$ has log canonical threshold $< 3/4$. Every other GIT polystable curve has lct $\geq 3/4$, so by the previous result, the pair $(\mathbb{P}^2, cD)$ is K-polystable for all $c \in (0, 3/4)$.

As the double conic cannot be K-semistable at $c = 3/4$, we must 'cross a wall' somewhere when increasing the coefficient where the pair $(\mathbb{P}^2, c(xy-z^2)^2)$ destabilizes and gets replaced by something else. In fact, for $E = (xy - z^2)$, one can compute that $\delta_{\mathbb{P}^2, c(xy-z^2)^2}(E) \geq 1$ if and only if $c \leq 3/8$, so the pair could be K-semistable if and only if $c \leq 3/8$. This is in fact true: if D is the double conic, $(\mathbb{P}^2, cD)$ is K-semistable if and only if $c < 3/8$. And, when $c = \frac{3}{8}$, we can do the following. Consider a family D of smooth quartic curves degenerating to the double conic, inside $X = \mathbb{P}^2 \times \mathbb{A}^1$. In X, blow up the conic. This produces a threefold Y with exceptional divisor $E \cong \mathbb{F}_4$. Let D_Y be the strict transform of D in Y. Now, the surface that was the original central fiber of X is contractible, and we can contract it to produce a family Z of $\mathbb{P}^2$ degenerating to $\mathbb{P}(1,1,4)$. As we cross the wall at $c = \frac{3}{8}$, we can verify that the new central fiber $(\mathbb{P}(1,1,4), (c+\epsilon)D')$ is K-semistable. This is illustrated in Figure 2.

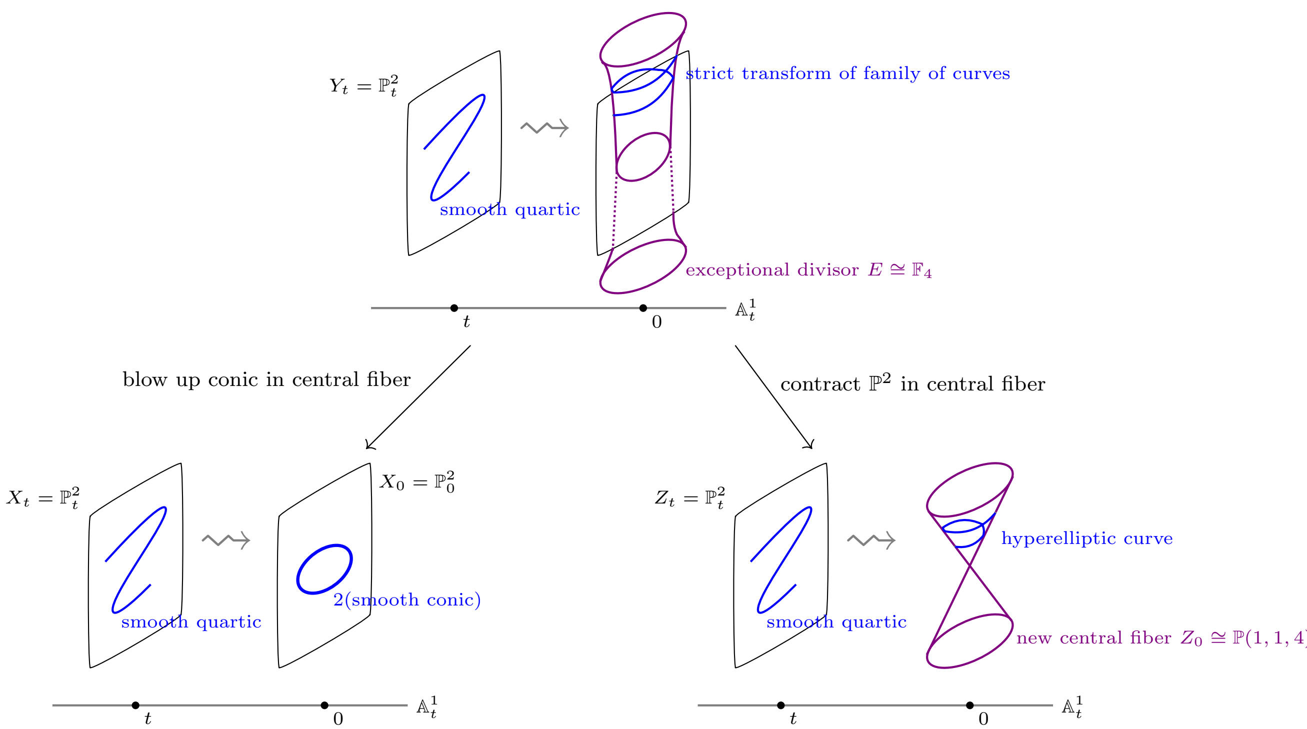

FIGURE 2. Replacement of the double conic.

After we cross this wall, we have curves on both $\mathbb{P}^2$ or $\mathbb{P}(1,1,4)$ appearing. In fact, for quartic curves, this is the only wall crossing in the K-moduli spaces!

This behavior generalizes to a wall-crossing phenomenon in general [ADL19, Zho22].

Theorem 5.7. *Assume $D \sim_{\mathbb{Q}} -rK_X$, with X Fano, and $c \in (0, \min\{1, r^{-1}\})$ or $c \in (\min\{1, r^{-1}\}, 1))$. Then, there are finitely many rational numbers*

$$0 = c_0 < c_1 < c_2 < \cdots < c_k = \min\{1, r^{-1}\} < \cdots < c_n = 1$$

such that the associated moduli spaces do not change for $c \in (c_i, c_{i+1})$. For each $1 \le i \le n-1$, $i \neq k$ and $0 < \epsilon \ll 1$, we have the following diagram

$$\begin{array}{ccccc} \mathcal{M}_{c_i-\epsilon} & \longrightarrow & \mathcal{M}_{c_i} & \longleftarrow & \mathcal{M}_{c_i+\epsilon} \\ \downarrow & & \downarrow & & \downarrow \\ M_{c_i-\epsilon} & \longrightarrow & M_{c_i} & \longleftarrow & M_{c_i+\epsilon} \end{array}$$

where the vertical arrows are the good moduli space morphisms, the arrows in the first row are open immersions, and the arrows in the bottom row are projective morphisms.

For $i = k$, we also have this diagram in the plane curve case.

There are similar (although more complicated) results for the KSB(A) moduli spaces [ABIP23].

We can compute the entire wall-crossing picture for quartic plane curves and find only two more walls. All wall crossings for quartics are summarized in Figure 3.

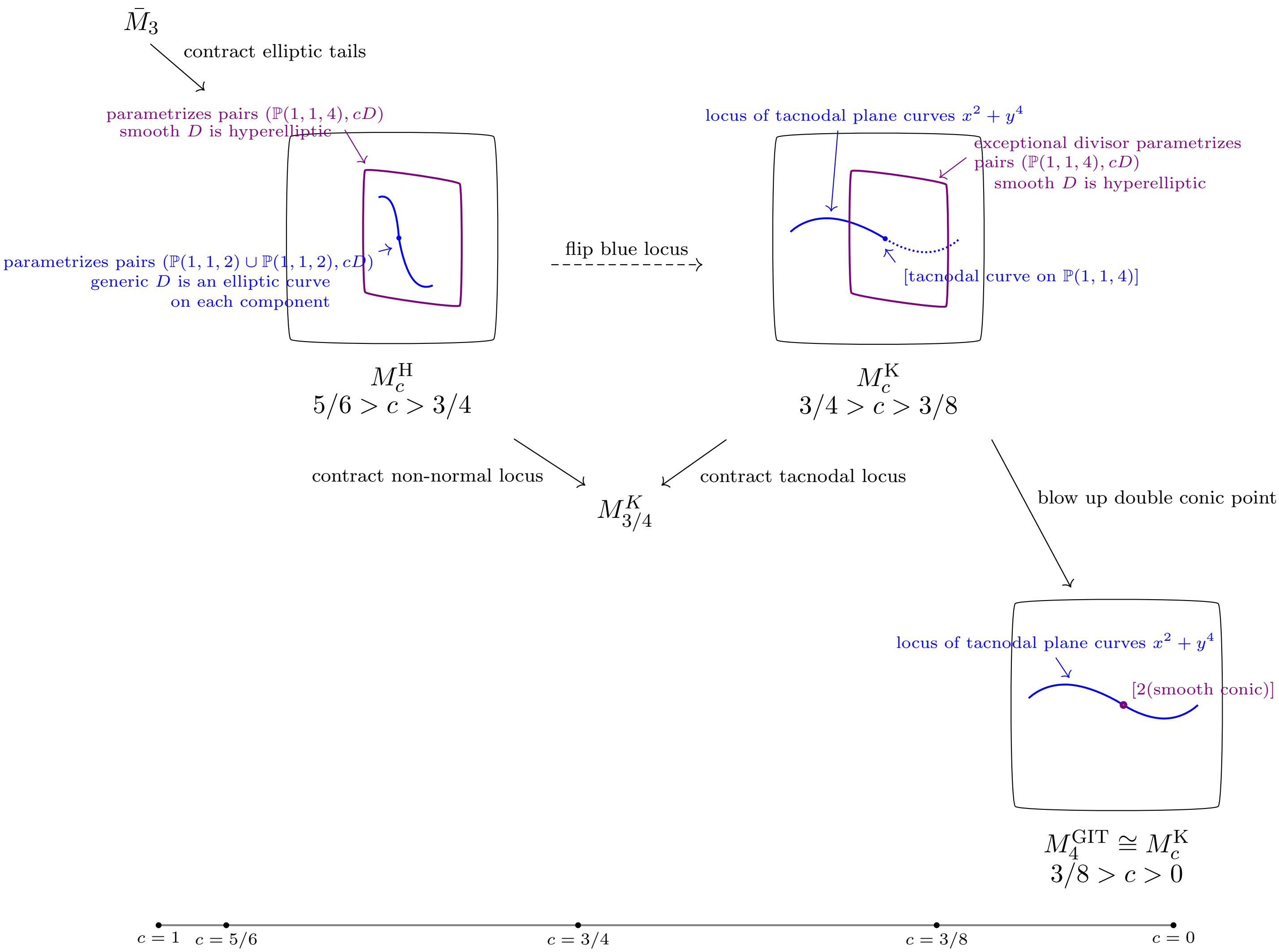

FIGURE 3. Wall crossings for moduli of quartic curves.

Note the picture includes values of $c \geq \frac{3}{4}$ in the log Calabi Yau and canonically polarized region. The left side of the picture was worked out by Hassett. Note for $c > \frac{5}{6}$, we write $\overline{M}_3$ instead of the moduli space of pairs $(\mathbb{P}^2, cD)$ and their degenerations; it is true in this case that for $c \in (\frac{5}{6}, 1]$, the moduli spaces are isomorphic, and at $c = 1$ the forgetful map $(X, D) \to D$ is also an isomorphism. In particular, the moduli space of pairs in this range is isomorphic to the moduli space of stable genus 3 curves. Finally, at $c = \frac{3}{4}$, there does indeed exist a 'log Calabi Yau' moduli space but we will not go into detail on that here.

To summarize the general phenomena in this example, we see that we can interpolate between several different moduli spaces using this idea of wall-crossing. This has had several applications so far and we expect many more in general!

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